CO 250 — Fall 2024: Class Notes

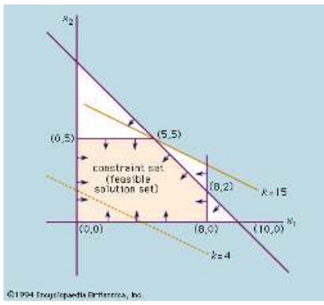
Jiucheng Zang

January 8th 2024

Contents

I. What's Optimization?

- Abstractly: want to
  - min/max some function f
  - over a subset A of n-dimensional reals
- Generally a very difficult question
- Here: we look at several important special cases that we understand better



i. Shortest Path Problem

- Given: two locations s and t in a road network
- Goal: find a "shortest" route connecting s ant t
- Linear programming and duality

ii. Traveling Sales Person Problem

- Given: a set of cities and distances between them
- Goal: find the shortest possible route that visits each city exactly once and returns to the origin city
- NP-hard problem
- Can be modelled as an integer program

II. Modeling Example

i. Ingredients of a Math Model

Production: WaterTech Production

WaterTech produces 4 products $\mathbb{P} = \{1, 2, 3, 4\}$ form the following resources:

- Time on two machines,
- Skilled and unskilled labor

The following table gives precise requirements:

Product	Machine 1	Machine 2	Skilled labor	Unskilled labor	Unit sale price
1	11	4	8	7	300
2	7	6	5	8	260
3	6	5	5	7	220
4	5	4	6	4	180

Restrictions:

- WaterTech has available 700h of machine 1, 500h of machine 2
- Can purchase 600h of skilled labor at \$8 per hour, and at most 650h of unskilled labor at \$6 per hour

Objective: How much of each product should WaterTech produce in order to maximize profit?

- Decision Variables: Capture unknown information
- Constraints: Describe what assignments to variables are feasible.
- Objective Function: A function of the variables that we would like to maximize/minimize.

1. Decision Variables

Introduce variable x_i for number of units of product i produced.

2. Constraints

For convenience, we can introduce: y_s, y_u: number of hours of skilled/unskilled labor to purchase.

Example:

Production plan described by assignment may not use more than 700h of time on machine 1.

$$\Rightarrow 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700$$

For revenue from sales:

$$300x_1 + 260x_2 + 220x_3 + 180x_4$$

For cost of labor:

$$8y_s + 6y_u$$

Objective function:

$$\max \quad 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u$$

Entire model:

$$\begin{aligned} \max \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\ \text{s.t.} \quad & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\ & 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\ & 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\ & 7x_1 + 8x_2 + 7x_3 + 4x_4 \leq y_u \\ & y_s \leq 600 \\ & y_u \leq 650 \\ & x_i, y_s, y_u \geq 0 \end{aligned}$$

ii. LP Models

We consider optimization problems of the following form:

$$\min\{f(x) : g_i(x) \leq b_i, (1 \leq i \leq m), x \in \mathbb{R}^n\}$$

- $n, m \in \mathbb{N}$
- $b_i \in \mathbb{R}$
- $f, g_i : \mathbb{R}^n \to \mathbb{R}$

Notes: all functions are affine in this class.

Def :
A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = a^T x + \beta$ for $a \in \mathbb{R}^n, \beta \in \mathbb{R}$.
It is linear if in addition $\beta = 0$.

Example:

1. $f(x) = 2x_1 + 3x_2 - x_3 + 7$ (affine, but not linear)
2. $f(x) = -3x_1 + 5x_3$ (linear)
3. $f(x) = 5x - 3\cos(x) + \sqrt{x}$ (not affine and not linear)

Def :
The optimization problem

$$\min\{f(x) : g_i(x) \leq b_i, \forall 1 \leq i \leq m, m \in \mathbb{R}^n\} \quad (P)$$

is called a linear program if is affine and g_i is a finite number of linear functions.

- Instead of set notation, we often write LPs more verbosely
- May use max instead of min
- Often give non-negativity constraints separately
- Sometimes replace subject to by s.t.
- We often write $x \geq \mathbb{0}$ as a short for all variables are non-negative. Example:

$$\begin{aligned} \min \quad & -x_1 - 2x_2 - x_3 \\ \text{s.t.} \quad & 2x_1 + x_3 \leq 3 \\ & x_1 + 2x_2 = 2 \\ & x \geq \mathbb{0} \end{aligned}$$

- Second mathematical program is not an LP. Three reasons:

- Dividing by variables is not allowed
- Cannot have strict inequalities
- Must have the finite number of constraints
  Example:

$$\begin{aligned} \max \quad & -1/x_1 - x_3 \\ \text{s.t.} \quad & 2x_1 + x_3 < 3 \\ & x_1 + \alpha x_2 = 2 \quad \forall \alpha \in \mathbb{R} \end{aligned}$$

iii. Interlude

Given: m by n matrix $A, b \in \mathbb{R}^m$

Suppose: $Ax = b$ has no solution

Goal: Find a vector x that comes close to being feasible;
i.e., a vector that minimizes

$$\sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j - b_i \right|$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} y_i \quad (P) \\ \text{s.t.} \quad & \left| \sum_{j=1}^{n} a_{ij} x_j - b_i \right| \le y_i \\ & \forall i \in \{1, \ldots, m\} \end{aligned}$$

(This is not an LP, since a linear program with absolute values, but we can write it as an LP)

1. Absolute value

$a \in \mathbb{R}$, find an LP about optimal value is $|a|$.

Assume y decision variable. What optimal solution to have $y^* = |a|$?

Claim: Function optimal solution y^* to (P) satisfies $y^* = |a|$.

$\underline{Pf.}$

Feasibility $\Rightarrow y^* \ge a,\ y^* \ge -a$ (Present $y^* \ge |a|$). Know $|a| \in \{a, -a\}$

Suppose for contradiction that $y^* > |a|$. There is an $\epsilon > 0$ s.t. $y^* = |a| + \epsilon$.

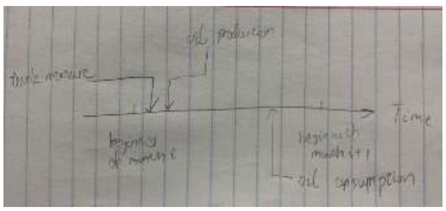
Then $\overline{y} = y^* - \epsilon$ is also feasible for (P) and has smaller objective value, contradiction. □

iv. Multiperiod Model

Month:	1	2	3	4
Demand:	5000	8000	9000	6000
Price:	.75	.72	.92	.90

Goal: "How much oil should company produce in each month to satisfy demand?"

Assumption:



Tank restriction:

- Initially contains 2 kL of oil.
- Can hold no more than 4 kL.

Decision variables: p_i # liters of oil produced in month i.

Some other variables(Not needed but useful): t_i # liters of oil in tank in month i.

When is p_i, t feasible?

- When we have enough oil to satisfy demand.

$$t_i + p_i \geq d_i, \forall i, t_1 = 2000$$

$$t_i + p_i = d_i + t_{i+1}, \forall i \in \{1,2,3\} (b_i) \text{(balance constraints)}$$

- Oil in tank does not exceed 4 kL.

$$t_i \leq 4000, \forall i \in \{2,3,4\}$$

- Non-negativity constraints.

$$p_i \geq 0, t_i \geq 0, \forall i \in \{1,2,3,4\}$$

Formal the LP:

$$\begin{aligned} \min \quad & \sum_{i=1}^{4} p_i \cdot c_i \\ \text{s.t.} \quad & (b_i)\ \forall i \in \{1,\dots,4\} \\ & t_i \leq 4000\ \forall i \in \{2,\dots,5\} \\ & p \geq 0, t \geq 0 \\ & t_i \geq 0\ \forall i \in \{1,\dots,5\} \\ & t_1 = 2000 \end{aligned}$$

Let (p^*, t^*) optimal solution

Could it be that $t^* > 0$ (i.e $t_i^* > 0$ for some i)?

NO! Suppose that $t_i^* \geq \epsilon > 0\ \forall i$

$\overline{\phi}$ constructed from $p^*|$ Suppose $p_{i'}^* > 0$ for some i'.

$$\overline{p_i} = \begin{cases} p_i^* & i \neq i' \\ p_{i'}^* - \epsilon & i = i'\ \overline{\epsilon} = \min\{p_{i'}^*, \epsilon\} \end{cases}$$

Define: $\overline{t_i} = \overline{t_i - 1} + \overline{p_{i-1}} - d_{i-1} \forall i$

Claim: $(\overline{p}, \overline{t})$ is feasible and has a smaller objective value.

Pf.

$$\begin{aligned} \sum_{i=1}^{4} \overline{p_i} c_i &= \sum_{i=1}^{4} p_i^* c_i - \overline{\epsilon} c_{i'} \\ &< \sum_{i=1}^{4} p_i^* c_i \end{aligned}$$

□

v. Integer Programming

Linear programming: Have $I \subseteq \{1, \ldots, n\}$, $x_i \geq 0\ \forall i \in I$.

$$\begin{aligned} \min / \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0\ \forall i \in S, S \subseteq \{1, \ldots, n\} \\ & A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n \\ & x_i \text{ integer } \forall i \in \{1, \ldots, n\} \end{aligned}$$

Notes: The resulting mathematical program is a mixed integer program (MIP).

Same variables are fractional, others are integer.

Knapsack items $i \in \{1 \ldots n\}$

- Each item i has a value $v_i > 0$ and a weight $w_i > 0$.
- Also have weight limit $B > 0$.

Problem: Pick $S' \subseteq \{1, \dots, n\}$ s.t. $\sum_{i \in S'} v_i$ is maximum and $w(S') = \sum_{i \in S'}^{w_i} \leq B$.

Need to pick at most n_i copies of item $i \in \underbrace{[n]}_{\{} 1, \dots, n\}$.

Decision variables: x_i # items of types i $\forall i \in [n]$.

$$\begin{aligned} \max \quad & \sum_{i \in [n]} x_i v_i \\ \text{s.t.} \quad & \sum_i w_i x_i \leq B \\ & x_i \in \{0, \dots, n_i\} \\ & x_i \in \mathbb{Z}, x_i \geq 0 \ \forall i \in [n] \end{aligned}$$

Knapsacks are weekly NP-hard.

Extra: $x_i \in \{0, 1\}$, $x_i \in \{0,1\} \forall i \in [n]$.

Can choose item of type $i = 2$ only if we also choose at least one item of type $i = 3$.

idea: create binary variable y

$\Big\{ y = 1 \Rightarrow x_2 > 0 y = 0 \Rightarrow x_2 = 0$

$x_3 \leq M \cdot y$

Knapsack items $I \subseteq \mathbb{N}, \forall i \in I$ have weight w_i, value v_i

Have capacity $B > 0$.

Want: find $S \subseteq I$ s.t. $W(S) \leq B$ and $V(S')$ max.

Can pick no more than $n_i \in \mathbb{N}$ items of $i \in I$.

$$\begin{aligned} \max \quad & \sum_{i \in I} v_i x_i \\ \text{s.t.} \quad & \sum_{i \in I} w_i x_i \leq B \\ & 0 \leq x_i \leq n_i, \forall i \in I \\ & x_i \text{ integer } \forall i \in I \end{aligned}$$

Want: find $x_i \in \{0, \dots, n_i\}$ s.t. $\sum_{i \in I} w_i x_i \leq B$ and $\sum_{i \in I} v_i x_i$ maximum.

Extra: 0-1 Knapsack $x_3 > 0$ only if $x_2 > 0$.

$$x_3 \leq M \cdot x_2$$

$x_2 = 0 \Rightarrow$ any feasible solution has $x_3 = 0$.

Want x feasible for original IP and $x_2 > 0 \Rightarrow$ x feasible for new formulation.

For this to be satisfied, M needs to be sufficiently large.

Example:

$M = m_3$
That: x is feasible for original IP $\Rightarrow$ (z) $\Rightarrow x_3 \leq m_3$.
If $x_2 > 0$, then the rhs is $\geq m_3 \Rightarrow$ x satisfied

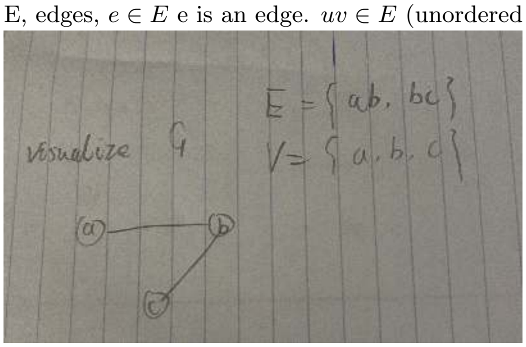
Tips: For solving an IP
It is good practice to choose M in big-M constraints as small as possible.
Depends on the problem, $x_3 \leq \left\lfloor \frac{B}{w_3} \right\rfloor \Rightarrow M = \min\left\{n_3, \left\lfloor \frac{B}{w_3} \right\rfloor\right\}$

Valid constraints:
A constraint $a^T x \leq b$ is valid for $S \subseteq \mathbb{R}^n$ if $x \in S \Rightarrow a^T x \leq b$.
S' = {solutions x to IP with $x_3 > 0$ only if $x_2 > 0$}.
Variants of Knapsack problem: $i \in I$ width w_i, length l_i, value $v_i \in \mathbb{R}$.

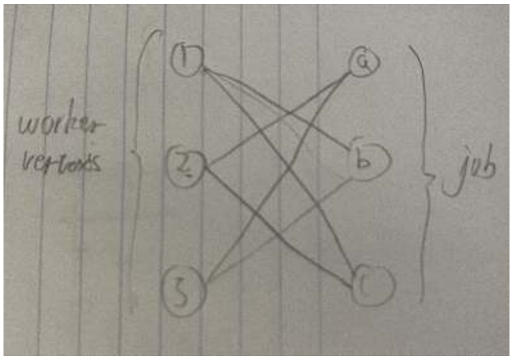
III. Graphs

i. Graphs Formal Definition

Graphs $G = (V, E)$
V, vertices, $v \in V$ v is a vertex.
E, edges, $e \in E$ e is an edge. $uv \in E$ (unordered pair).



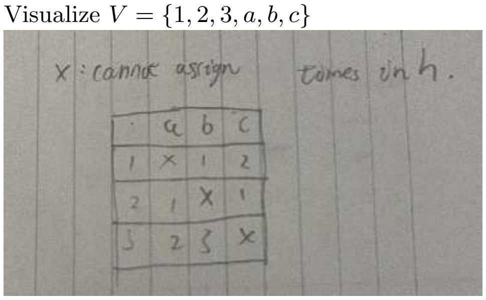
Example application have jobs $J = \{a, b, c\}$, workers $W = \{1, 2, 3\}$.
Need to do jobs in J. Assign jobs in J to workers in W.
Not every job can be done by every worker.
Workers take time to accomplish job.



Assign each worker to a unique job $j \in J$.
s.t.

- no worker is assigned to a job they cannot complete
- minimize total time it takes to finish all jobs

Visualize $V = \{1, 2, 3, a, b, c\}$



	a	b	c
1	X	1	2
2	1	X	1
3	2	3	X

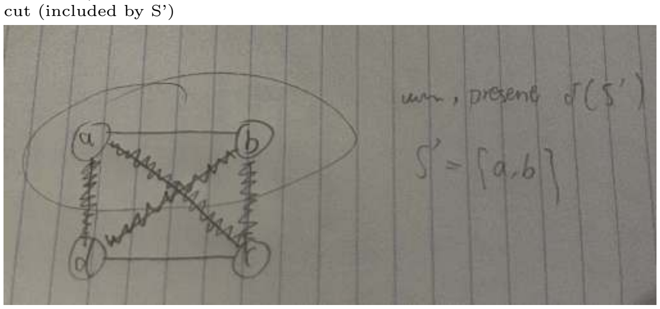
Add an edge connecting w and j, if worker w can do job j.
t_{ij} time it takes worker i to do job j.

ii. Cuts

$G = (V, E)$, $S' \subseteq V$.

Cut: $S' \subseteq V$ s.t. $\delta(S')$ is a cut.

$$\underbrace{\delta(S')}_{\text{cut (included by S')}} = \{uv \in E : u \in S', v \notin S'\}$$

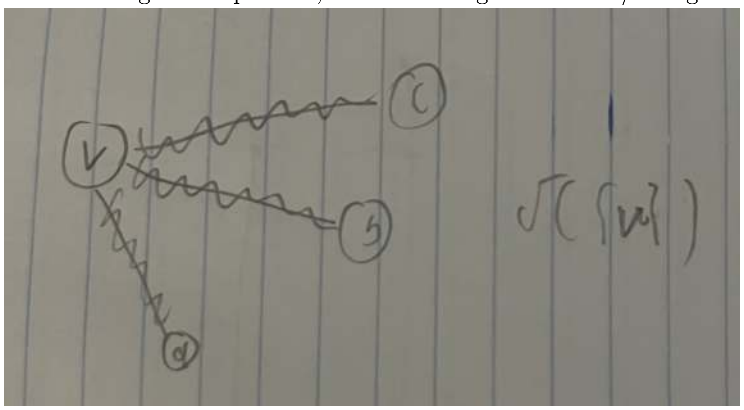


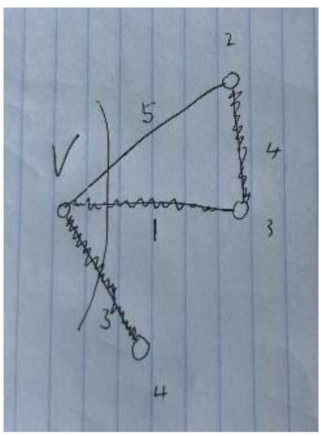
Find assignment of jobs to workers s.t. each $w \in W$ has unique assigned job.
Example:

> possible assignment $b \to 1, a \to 3, c \to 2$.
> Corresponds to edge $M = \{b1, a3, c2\}$.

$v \in V$, $\delta(\{v\})$ = set of edges with one endpoint in V.

Assignment of jobs to workers is feasible, δ the corresponding edge set M has at most one edge incident to any vertex.
$M \subseteq E$ it called a matching if $\delta(\{v\})$ has at most one edge from M. $\forall v \in V$.
Workers assignment question, Find matching in Workers / Assignment graph.





$M \subseteq E$ is a matching if $\forall ux, wx \in M$, $uv \neq wx$, $\{u,v\} \cap \{w,x\} = \emptyset$. (Not two edge share an endpoint)

Alternative: $\delta(v) = \{uv \in E : v \in \{u,v\}\}$

The cut indicates v.

M is a matching if

$$\forall v. |\delta(v) \cap M| \leq 1$$

δv has at most one element from M.

We can use that to write matching as MIP.

$$xe = \begin{cases} 1 : \text{edge e is in M} \\ 0 : \text{otherwise} \end{cases} \quad , \forall e \in E$$

Constraints:

$$\sum_{e\in\delta(v)} x_e \leq 1, \forall v \in V$$

Want:

Find $M \subseteq E$ matching s.st. $\sum_{e\in M} w_e$ is maximized.

We can formulate this as a MIP.

$$\begin{aligned} \max \quad & \sum_{e\in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e\in\delta(v)} x_e \leq 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E \end{aligned}$$

Example:

$$\begin{aligned} \max \quad & \begin{pmatrix} 5 & 1 & 3 & 4 \end{pmatrix} x \\ \text{s.t.} \quad & \begin{pmatrix} \text{V/E} & 12 & 13 & 14 & 23 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{pmatrix} x \leq \mathbb{1} \end{aligned}$$

$x \in \mathbb{R}^4, x \geq 0,$ x integer

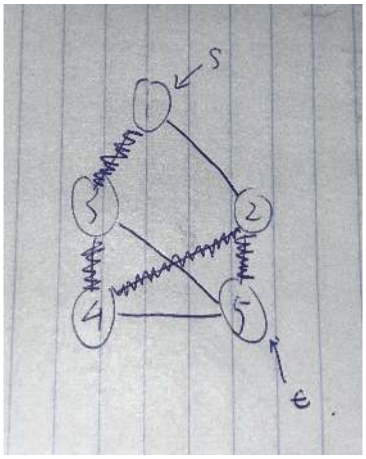
Note: IP, in general are hard to solve (NP-hard).

On the other hand matching have very fast algorithms.
Matching have nice properties, Let (P) the LP obtained from (IP) by deleting "integer".
IP (P) has integer optimal solutions.

iii. Path

$P = v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$ is a path in G (From s to t)
s.t.

1) $v_iv_{i+1} \in E, \forall 1 \leq i \leq k-1$
2) $v_i \neq v_j, \forall i \neq j$



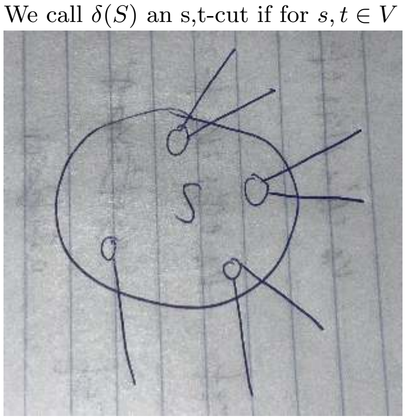
1-5 path is {13,34,42,25}
P is an s, t-path if p starts in **s** and ends in **t**. $c \in \mathbb{R}^E_{\geq 0}$ c_e is the length of edge $e \in E$.
Goal: find s, t-path p of minimum length $c(p) = \sum_{e \in p} c_e$

Detour
$\delta(v)$ = cut induced by v.

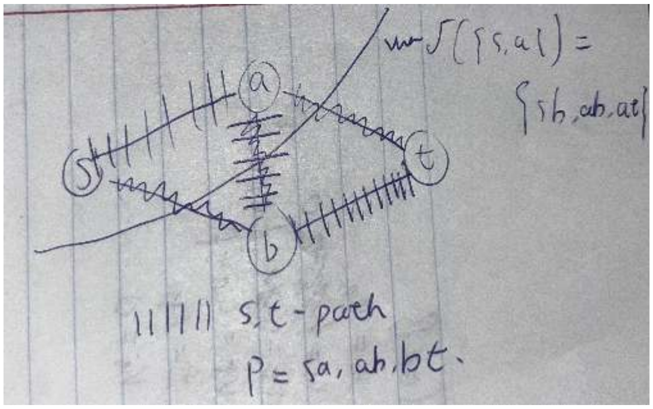
Define $\delta(S)$ set of all edges with exactly one end in S.

$$= \{ \underbrace{uv}_{\text{unordered pair}} \in E : u \in S, v \notin S\}$$

We call $\delta(S)$ an s,t-cut if for $s, t \in V$ if $s \in S$ and $t \notin S$.



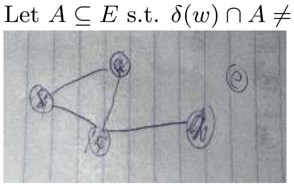
Remark: If $A \subseteq E$, contains at least one edge from all s, t-cut, then A has an s, t-path.



(ab line is in the set of $\delta(s, a)$)

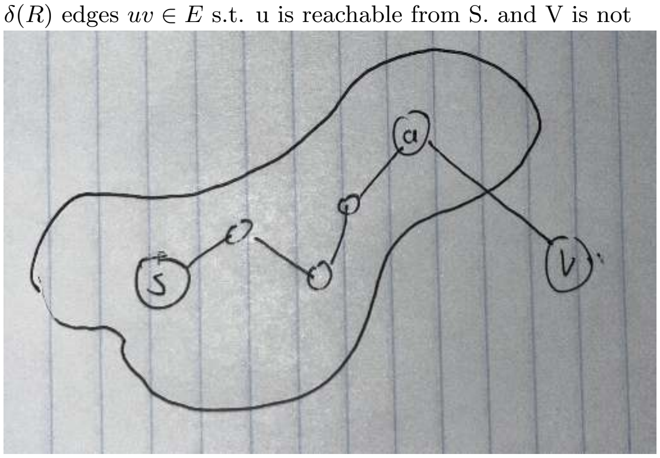
Pf.

Let $A \subseteq E$ s.t. $\delta(w) \cap A \neq \emptyset, \forall$ s,t-cut $\delta(w)$.



By contradiction assume that A contains no s, t-path.

Define $R = \{u \in V$ there is an s,t-path with edges in A$\}$
Observe that $t \notin R$
$\delta(R)$ edges $uv \in E$ s.t. u is reachable from S. and V is not



$\Rightarrow \exists$ s,t-path in A.

$\delta(R)$ is an s,t-cut
By assumption $A \cap \delta(R) \neq \emptyset$
Let $qr \in A \cap \delta(R), q \in R, r \notin R$

Assumption s-t-path in A $\Rightarrow$ q is reachable from S.
$\Rightarrow r$ is reachable from S.
$\Rightarrow r \in R \Rightarrow$ contradiction. $\square$

We can write IP for shortest path problem **Decision variables:**

$$x_e = \begin{cases} 1 : \text{edge e is in the path} \\ 0 : \text{otherwise} \end{cases} \quad , \forall e \in E$$

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(W)} x_e \geq 1, \forall \text{ s,t-cut } \delta(W) \\ & x_e \geq 0, \forall e \in E \end{aligned}$$

Let x^* be an optimal solution

$$p = \{e \in E : x_e^* \geq 1\}$$

Is p the set pf edges of a path?
If $c_e > 0$ $\forall e \in E$, then yes, p is the edge set of a path.

IV. Solving Linear Programs

(IP)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x_j \in \{0, 1\} \quad \forall j \end{aligned}$$

(LP)

$$\begin{aligned} \max / \min \quad & h(x) \\ \text{s.t.} \quad & g_i(x) = b_i \quad \forall i \\ & \text{h, g: non-linear functions} \end{aligned}$$

Note: IP harder than LP, because the feasible region is not convex.

Linear Programs Solution

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

1. $\bar{x} \in \mathbb{R}^n : A\bar{x} \leq b, \bar{x} \geq 0$ is a feasible solution.

   If exists feasible solution for (P) $\Rightarrow$ (P) is feasible.

2. Suppose that for any $M \in \mathbb{R}$ $\exists \bar{x}$ feasible for (P) s.t. $c^T\bar{x} \geq M$.
   Then, (P) is unbounded.
3. A feasible solution $\bar{x}$ for (P) is optimal
   If $\nexists \hat{x}$ feasible for (P) with $c^T\hat{x} < c^T\bar{x}$.

Q: If a given mathematical program is feasible and bounded, does it have an optimal solution?

No, because it's not linear programming, it's a general mathematical program.

Example:

$$\begin{aligned} \max \quad & x \\ \text{s.t.} \quad & x \leq 1 \end{aligned}$$

- (P) is feasible,
- (P) is bounded $x \leq 1$,

No optimal solution, $\bar{x}$ feasible

$$\Rightarrow \hat{x} = \frac{\bar{x}+1}{2} < 1, \quad \hat{x} > \bar{x}$$

LPs are special any LP satisfies one of the following conditions:

- It has an optimal solution
- It is infeasible
- It is unbounded

i. Certificates Infeasibility

(P)

$$\begin{aligned} \max \quad & (3,4,-1,2)x \\ \text{s.t.} \quad & \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

Claim: (P) is infeasible.
If $\bar{x}$ satisfies (P), then $\bar{x}$ also satisfies linear combinations of constraints.

$$[3x_1 - 2x_2 - 6x_3 + 7x_4 = 6] \cdot a$$
$$[2x_1 - x_2 - 2x_3 + 4x_4 = 2] \cdot b$$

$$(3a+2b)x_1 + (-2a-b)x_2 + (-6a-2b)x_3 + (7a+4b)x_4 = 6a+2b$$

In short, $y = (a,b)^T$

$\bar{x}$ feasible for (P) also satisfies $y^T A\bar{x} = y^T b$

Choose $y = (1,2)^T$
Then $y^T A\bar{x} = y^T b \Rightarrow \begin{pmatrix}1 & 0 & 2 & 1\end{pmatrix} x = -2$
Any feasible $\bar{x}$ needs to satisfy this!
But $x \geq 0$, so it's infeasible.

In general:
Proposition:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Has no feasible solution if $\exists y \in \mathbb{R}^m$ s.t. $y^T A \geq 0$ and $y^T b < 0$.

1. Strong Farkal's Lemma

$Ax = b, x \geq 0$ has no solution (iff) $\exists y \cdot y^T A \geq 0, y^T b < 0$
[Next week prove this]

ii. Proving optimality

$$\begin{aligned} \max \quad & (-1,-4,0,0)x + 4 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

Claim: $\bar{x} = (0,0,4,5)^T$ is optimal.
Pf.
$\bar{x}$ is feasible, $c^T\bar{x} = 4$.
Let $\hat{x}$ be any other feasible solutions.

$$c^T\hat{x} = 4 + (-\hat{x}_1 - 4\hat{x}_2) \leq 4 = c^T\bar{x}$$

Since $\hat{x} \geq 0$, then this is the optimal solution. □

iii. Certificates of Unboundedness

$$\begin{aligned} \max \quad & (-1,0,0,1)x \\ \text{s.t.} \quad & \begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

Idea: define family $\{x^t\}_{t\in\mathbb{N}}$ of feasible solutions.

$$\lim_{t\to\infty} \left(-1,0,0,1\right) x^t = \infty$$

LP is feasible: $\bar{x} = (0,1,1,0)^T$ is feasible

$$x^t = \bar{x} + t \cdot r \quad r = (1,0,1,2)^T$$

$$\begin{aligned} Ax^t &= A\bar{x} + t \cdot Ar \\ &= b + t \cdot 0 = b, \quad \bar{x} + tr \geq 0 \quad \text{feasible} \end{aligned}$$

Objective function:

$$\begin{aligned} & c^T x^t (-1,0,0,1) \cdot x^t \\ & (-1,0,0,1) \cdot \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t \cdot (-1,0,0,1) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \\ & = 1 + t \end{aligned}$$

If $t \to \infty$, then $c^T x^t \to \infty$

General:
(P)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

(P) is unbounded if

- $\exists \bar{x} \leq 0, A\bar{x} = b$
- $\exists \underbrace{r}_{ray}$, s.t., $Ar \geq 0, c^T r > 0$

iv. SEF(Standard Equality Form)

1. $\max LP$
2. every constraint is an equality
3. every variable is non-negative

Example:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Example:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 && = 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Idea: State an algorithm for solving LPs in SEF.
If we are given an LP not in SEF then constitute equivalent LP in SEF.

Equivalent

LPs (P) and (P') are equivalent if

1. P infeasible iff p' infeasible
2. P unbounded iff P' unbounded
3. That is a map between feasible solutions to P and P'

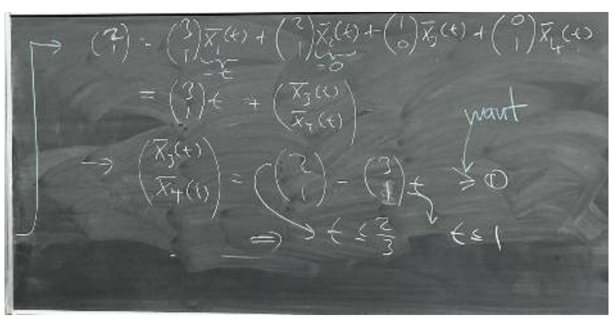
Theorem :

Every LP (P) has an equivalent LP in SEF.
Example:

$$\begin{aligned} \max \quad & (4, 3, 0, 0)x + 7 \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

Trivial solution: $x = \begin{pmatrix} 0 & 0 & 2 & 1 \end{pmatrix}^T$, with objective value 7.
Pick one of x_1, x_2 and by increasing it. $x_1 = t$New value > 0 and $x_2 = 0$. $\Rightarrow$ new solution, $\bar{x}(t) = (t, 0, ?, ?)^T$.
To get the new solution:
$\bar{x}(t)$ needs to be feasible

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \bar{(x)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\bar{x}(t) \geq 0$$



Then we can choose any $t \leq \frac{2}{3}$ and have $\bar{x}(t) \geq 0$.
$\bar{x}(t)$ always satisfies $A\bar{x}(t) = b$.
So, choose $t = \frac{2}{3} \Rightarrow \bar{x}(\frac{2}{3}) = (\frac{2}{3}, 0, 0, \frac{1}{3})^T$.
Note $\bar{x}(\frac{2}{3})$ is not optimal.

LP was in canonical form:
Strategies: want solutions and corresponding LP in canonical form.

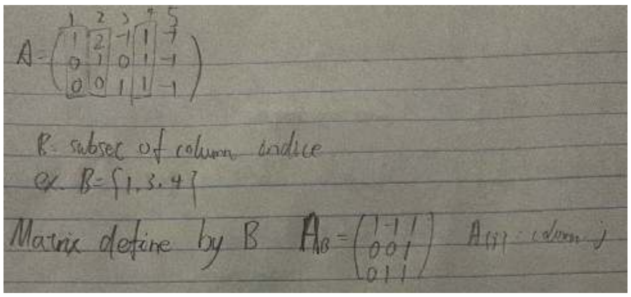
1. Find a feasible solution x
2. Rewrite LP in canonical form for x

3. If x is optimal, then we are done
4. If x is unbounded, then we are done
5. Find better solution than x, go to step 2

Totally three outcomes from SIMPLEX:

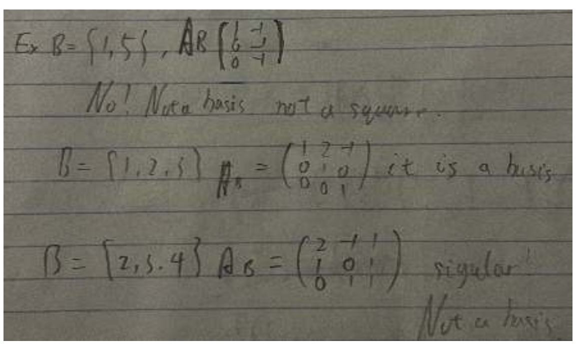
- Optimal
- Unbounded
- Have a better solution

v. Canonical form of matrix



Def :
If is an $m \times n$ matrix
$B \subseteq \{1, 2, \ldots, n\}$ is a basis of A if

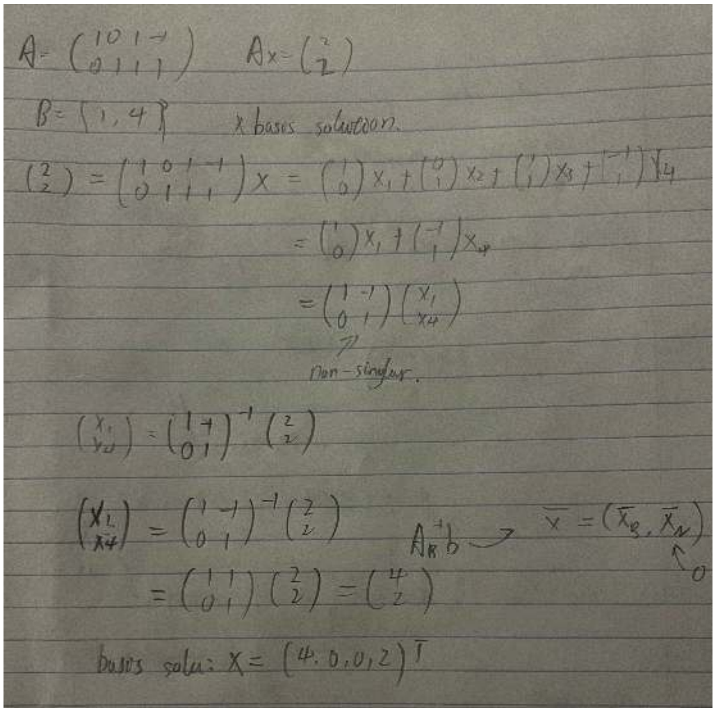
- A_B is square
- A_B is non-singular (Column is linearly independent)



Note: Not every matrix A have a basis, because there maybe a linearly dependent column.

Theorem :
For any A. The max # of linearly independent rows = max # of linearly independent columns.
Def :
x is a basic solution for basis B if

1. $A_x = b$
2. $x_j = 0$ for $j \notin B$



x is basic if x is a basic solution for some basis

Note: For every basis B, there is a unique basic solution, but a solution maybe basic for many bases.

Example:

$$\begin{pmatrix} 3 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$B = \{1,2\} = \{1,3\} = \dots \text{ Same basic solution } x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Remark: $Ax = b$, suppose that A has dependent rows. One of the two cases must happen:

1. infeasible system
2. can eliminate a row

Will assume A has full row route.

vi. Covert LP to SEF

Converting LPs to SEF

(P) $\min (-1, 2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \max (1, -2) x$

s.t. $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{matrix} \geq \\ = \end{matrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x_1 \geq 0$, x_2 free.

$x_2 = x_2^+ - x_2^-$

$x_2^+, x_2^- \geq 0$.

$\Updownarrow$

$\max (1, -2) x$

s.t. $\begin{pmatrix} 1 & 2 & -2 \\ -3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2^+ \\ x_2^- \end{pmatrix} \begin{matrix} \geq \\ = \end{matrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x_1 \geq 0$, $x_2^+, x_2^- \geq 0$

Side: $(1\ 2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1\ 2) \begin{pmatrix} x_1 \\ x_2^+ - x_2^- \end{pmatrix} = x_1 + 2x_2^+ - 2x_2^- = (1\ 2\ -2) \begin{pmatrix} x_1 \\ x_2^+ \\ x_2^- \end{pmatrix}$

$\Updownarrow$

Introduce new variable x_3.

(P') $\max (1, -2, 2, 0)$

$\begin{pmatrix} 1 & 2 & -2 & -1 \\ -3 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2^+ \\ x_2^- \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x_1 \geq 0$

$x_2^-, x_2^+ \geq 0$

$x_3 \geq 0$.

(P') is in SEF

Claim: There is a bijection between feasible solution to (P) and (P') preserves objective value.

(P) $\max c^T x$ basis $B \subseteq [n]$
s.t. $Ax = b$ (P) is in canonical form for B if
$x \geq 0$
1) $A_B = I$ indentity
2) $c_B = 0$
↑ coefficients of c coresponding to B

ex. $\max (0\ 0\ 2\ 4)\, x$
s.t. $\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$x \geq 0$

↑ canonical for $B = \{1, 2\}$

Suppose basis $B' \subseteq [n]$
$B' \neq B$
in ex. $B' = \{2, 3\}$
want equivalene LP (P')
that is in canonical form for B'

1) replace $Ax = b$ with $A'x = b'$ s.t. $A'_{B'} = I$

$A_{B'}^{-1} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 2 \end{pmatrix} x = A_{B'}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ⇐ solutions to this system are the same as those for $Ax=b$

⇓

$\begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

need rewrite objective function.
$c^T x = (0, 0, 2\, 4)\, x$
rewrite, $\bar{c}^T x + z$ s.t. $c^T x = \bar{c}^T x + z$
$\forall$ feasible x.

want $\bar{c}_B = 0$. Ax is feasible $\Longrightarrow Ax = b$ and $y^TAx = y^Tb$ $\forall y \in R^m$.

x feasible for $Ax = b \Rightarrow y^TAx = y^Tb$

$\iff y^Tb - y^TAx = 0$.

objective of x ← feasible.

$$c^Tx = c^Tx + y^T\underbrace{(b - Ax)}_{0} = \underbrace{(c^T - y^TA)}_{\bar{c}}x + y^Tb$$

want $\bar{c}_{B'} = 0$.

$$C^T_{B'} - y^TA_{B'} = 0 \iff y^T = A_{B'}^{-1} C^T_{B'}$$

$$y = C_B A_{B'}^{-T}$$

e.x. x feasible for

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow (y_1, y_2)\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = (y_1, y_2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$y = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \downarrow 2.$$

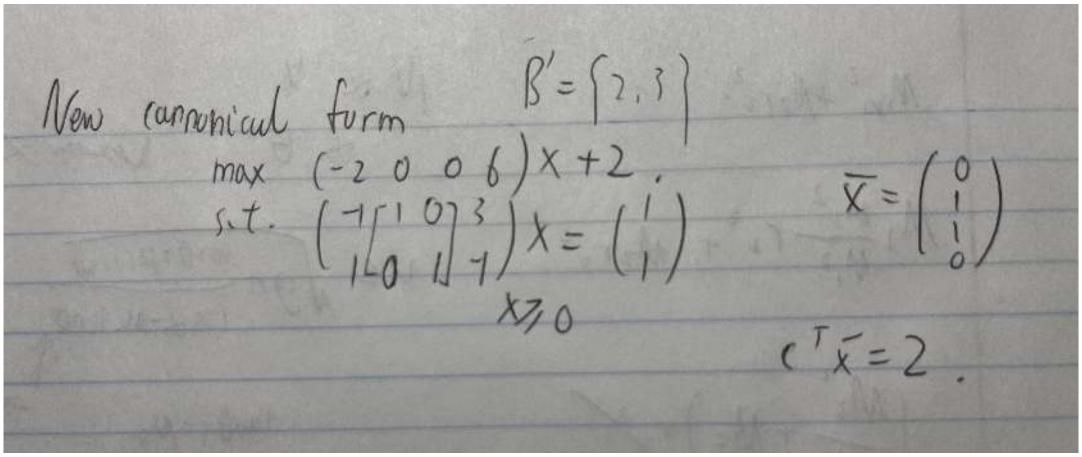
$$\Rightarrow (-2\ 0\ -2\ -2)\,x = 2$$

$$\Rightarrow 2 + (-2\ 0\ -2\ 2)\,x = 0.$$

objective value of x

$$(0\ 0\ 2\ 4)\,x = (0\ 0\ 2\ 4)\,x + \underbrace{(-2\ 0\ -2\ 2)\,x + 2}_{0}$$

$$= \underbrace{(-2\ 0\ 0\ 6)}_{\bar{c}^T}\,x + 2.$$



(P)

$$\begin{aligned} \max \quad & z = \begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix} x \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

(P) is in canonical form for $B = \{1, 2\}$

- $A_B = I$
- $C_B = 0$

$x = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}^T$ is the basic feasible solution.

$B' = \{2, 3\}$ write (P) in canonical form for B'

Idea: x satisfies $Ax = b \Rightarrow$ also satisfies $y^T A_x = y^T b$ for any $y \in \mathbb{R}^m \Leftrightarrow$ x satisfies $y^T(-Ax + b) = 0$

Example:

$$y = \begin{pmatrix} 2 \\ 0 \end{pmatrix}^T$$

$$\begin{aligned} \begin{pmatrix} 2 & 0 \end{pmatrix} \left(-\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) &= 0 \\ \begin{pmatrix} -2 & 0 & -2 & 2 \end{pmatrix} x + 2 &= 0 \end{aligned}$$

for any feasible x for (P)

$$\begin{aligned} \Rightarrow c^T x &= \begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix} x + \begin{pmatrix} -2 & 0 & -2 & 2 \end{pmatrix} + 2 \\ &= \begin{pmatrix} -2 & 0 & 0 & 6 \end{pmatrix} x + 2 \end{aligned}$$

(P)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

We choose basis B,
Then we can choose y s.t. $y^T A = c_B^T$

$$\begin{aligned} & y^T A = c_B^T \\ \Rightarrow & y^T = c_B^T A^{-1} \\ \Rightarrow & y = c_B (A^{-1})^T \end{aligned}$$

<u>Canonical form</u> For basis B. $y = A_B^{-T} c_B$

$$\begin{aligned} \max \quad & [c^T - y^T A]x + \underbrace{y^T b}_{\text{Value of basic solution}} \\ \textbf{s.t.} \quad & A_B^{-1} A_x = A_B^{-1} b \\ & x \geq 0 \end{aligned}$$

vii. Simplex

- Start with some feasible basic solution x for LP (P) in SEF, canonical form B.
- More from solution to solution, all basic feasible.
- Detect optionality, unboudedness $\Rightarrow$ certificate.

Note: Proof of termination of establishes fundamental theorem of LP.

(P)

$$\begin{aligned} \max \quad & \begin{pmatrix} 0 & 3 & 1 & 0 \end{pmatrix} x \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

$B = \{1, 4\}$, $x = \begin{pmatrix} 2 & 0 & 0 & 5 \end{pmatrix}^T$ $c^T x = 0$
$N = \{2, 3\}$, non-basic variables.
We don't know if x is optimal or not, may not be increasing.

<u>**Strategy:**</u> Provide $j \in N = [n] - B$ such that $c_j > 0$.
e.g. Choose $j = 2$
Attempt to increase $x_j = t \geq 0$, $x_{j'} = 0 \quad \forall j' \in N\{j\}$
$\Rightarrow x' = \begin{pmatrix} x_1' & t & 0 & x_4' \end{pmatrix}^T$
Need x' to be feasible

Example:

Iteration 1:

$$\begin{aligned} & \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} x^1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ \Leftrightarrow & \begin{pmatrix} x_1^1 \\ x_4^1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \geq 0 \\ \Rightarrow & t \leq \min\{1, 5\} \end{aligned}$$

Choose $t = 1$.
New solution. $x^1(t=1) = \begin{pmatrix} 0 & 1 & 0 & 4 \end{pmatrix}^T$
This is a basic feasible solution for $B' = \{2, 4\}$
Canonical form for new basis

$$A_{B'} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A_{B'}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$A_{B'}^{-T} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

$$C_{B'} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$y = A_{B'}^{-T} c_{B'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 \end{pmatrix}^T$$

$$\begin{aligned}
\max \quad & (c^T - y^T A)x + y^T b \\
& = (\begin{pmatrix} 0 & 3 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} & 0 \end{pmatrix} A)x + \begin{pmatrix} \frac{3}{2} & 0 \end{pmatrix} b \\
& = \begin{pmatrix} -\frac{3}{2} & 0 & 4 & 0 \end{pmatrix} x + 3 \\
\text{s.t.} \quad & A_{B'}^{-1} A x = A_{B'}^{-1} b \\
& = \begin{pmatrix} \frac{1}{2} & 1 & -1 & 0 \\ -\frac{1}{2} & 0 & 4 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\
& x \geq 0
\end{aligned}$$

Leaving variables: $B = \{1, 4\}$ old basis, $B' = \{2, 4\}$ new basis. $x' = (0, 1, 0, 4)^T$ is the new basic feasible solution. $c^T x' = 3$

Iteration 2: Since $c'_3 = 4 > 0$, we can increase the objective function value.

$$\Rightarrow x'' = \begin{pmatrix} 0 & x''_2 & t & x''_4 \end{pmatrix}^T, \quad t \geq 0$$

$$\begin{pmatrix} x''_2 \\ x''_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} - t \begin{pmatrix} -1 \\ 4 \end{pmatrix} \geq 0$$

$$-1 \leq t \leq 1$$

$x''(t=1) = \begin{pmatrix} 0 & 2 & 1 & 0 \end{pmatrix}^T$ is the new basic feasible solution.
This is a basic solution for $B'' = \{2, 3\}$

$$A_{B''} = \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}$$

$$A_{B''}^{-1} = \begin{pmatrix} 1 & \frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix}$$

$$A_{B''}^{-T} = \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$C_{B''} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$y = A_{B''}^{-T} c_{B''} = \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

New canonical form:

$$\begin{aligned}
\max \quad & \left(\begin{pmatrix}-\frac{3}{2} & 0 & 4 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\end{pmatrix} A\right)x + \left(\begin{pmatrix}0 & 1\end{pmatrix} b + 3\right) \\
& = \begin{pmatrix}-1 & 0 & 0 & -1\end{pmatrix} x + 7 \\
\text{s.t.} \quad & = A_{B''}^{-1} A x = A_{B''}^{-1} b \\
& = \begin{pmatrix}1 & \frac{1}{4} \\ 0 & \frac{1}{4}\end{pmatrix} \begin{pmatrix}\frac{1}{2} & 1 & -1 & 0 \\ -\frac{1}{2} & 0 & 4 & 1\end{pmatrix} x = \begin{pmatrix}1 & \frac{1}{4} \\ 0 & \frac{1}{4}\end{pmatrix} \begin{pmatrix}1 \\ 4\end{pmatrix} \\
& \begin{pmatrix}\frac{3}{8} & 1 & 0 & -\frac{1}{4} \\ -\frac{1}{8} & 0 & 1 & \frac{1}{4}\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix} \\
& x \geq 0
\end{aligned}$$

$x'' = \begin{pmatrix}0 & 2 & 1 & 0\end{pmatrix}^T$ is the new basic feasible solution. $c^T x'' = 7$

Why can we say that x'' is optimal? Do we have to check other basic?

(P) Original LP (P") Final canonical form.

We know (P) and (P") are equivalent $\Rightarrow$ for any feasible x for (P), x feasible for (P").

$$\underbrace{c^T x}_{\text{Original Objective}} = \underbrace{(c^T - y^T A_B'')}_{(-1,0,0,-1)} x + \underbrace{y^T b}_{7} \leq 7$$

<u>Slightly Modified example</u>

$$\begin{aligned}
\max \quad & \begin{pmatrix}-\frac{3}{2} & 0 & 4 & 0\end{pmatrix} x + 3 \\
\text{s.t.} \quad & \begin{pmatrix}\frac{1}{2} & 1 & -2 & 0 \\ -\frac{1}{2} & 0 & -1 & 1\end{pmatrix} x = \begin{pmatrix}1 \\ 4\end{pmatrix} \\
& x \geq 0
\end{aligned}$$

$x = \begin{pmatrix}0 & 1 & 0 & 4\end{pmatrix}^T$

Maybe not optimal because of $c_3 > 0$.

Try: $\bar{x} = \begin{pmatrix}0 & \bar{x}_2 & t & \bar{x}_4\end{pmatrix}^T$

Want to satisfy system of equations:

$$\begin{pmatrix}\bar{x}_2 \\ \bar{x}_4\end{pmatrix} = \begin{pmatrix}1 \\ 4\end{pmatrix} - t \begin{pmatrix}-2 \\ -1\end{pmatrix} \geq 0$$

Remains non-negative $\forall t \geq 0 \Rightarrow$ LP is unbounded.

Contract certificate of unsoundness:

$$\bar{x}(t) = \begin{pmatrix}0 & 1+2t & t & 4+t\end{pmatrix}^T$$

Feasible point $\bar{x} = \begin{pmatrix}0 & 1 & 0 & 4\end{pmatrix}^T$ need d s.t. $Ad = 0$ and $c^T d > 0$.

$d = \begin{pmatrix} 0 & 2 & 1 & 1 \end{pmatrix}^T \rightarrow Ad = 0 \quad c^T d > 4$

Simplex in General

(P)

$$\begin{aligned} \max \quad & c^T x + \bar{z} \\ \text{s.t.} \quad & Ax = b \quad A \in M_{m \times n}(\mathbb{R}) \\ & x \geq 0 \end{aligned}$$

Suppose (P) is in canonical form for $B \ldots A_B = I, c_B = 0$.

Notation: $x = \begin{pmatrix} \underbrace{x_B}_{\text{basic components}} & \underbrace{x_N}_{Non-basic components} \end{pmatrix}^T$ Also, $A = \begin{pmatrix} A_B & A_N \end{pmatrix}$

(P)

$$\begin{aligned} \max \quad & c_N^T x_N + \bar{z} \\ \text{s.t.} \quad & x_B + A_N x_N = b \\ & x \geq 0 \end{aligned}$$

Note:

$$\begin{aligned} c^T x &= (c_B, c_N)^T \\ \rightarrow c^T x &= c_N^T x_N \end{aligned}$$

Simplex Method:

1. Pick entering variable, $k \in N$ s.t. $c_k > 0$.
2. Want to investigate new solution

$$x' \quad x'_j = \begin{cases} t & j = k \\ 0 & j \in M \backslash \{k\} \end{cases}$$

$$\Rightarrow x'_B = b - tA_k \geq 0$$

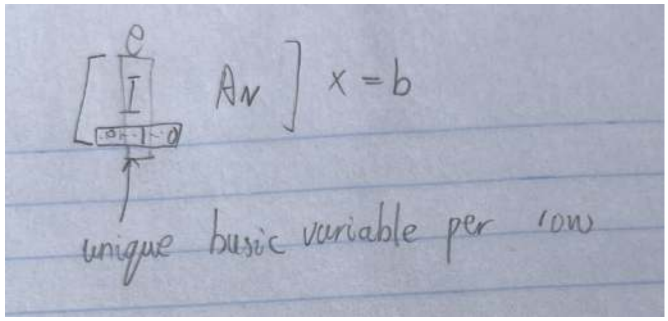
Proposition: IF $A_K \leq 0 \Rightarrow$ LP is unbounded.

Pf.

Assume $A_k \nleq 0 \Rightarrow b - tA_k \geq 0 \quad t \leq \min \left\{ \frac{b_i}{A_{ik}} \right\} \quad \forall i, A_{ik} > 0 \ (t \geq 0 \quad [b \geq 0])$

Choose t maximum s.t. $x'_B \geq 0 \Rightarrow$ LP is unbounded.

Let $r \in [m], t = \frac{b_r}{A_{rk}}$ corresponding to unique basic variable e (leaving variable).



$x_k \uparrow (t) \leftarrow$ entering variable.

$x_e \downarrow (t) \leftarrow$ leaving variable.

New basis: $B' = B \backslash \{e\} \cup \{k\}$ $\square$

Proposition: If simplex increases entering variable by $t > 0$ in every integration, then simplex terminates.

Pf.

Simplex enumerates basic solutions. $B_1 \to B_2 \to \dots \to B_k \to \dots$

unique: $x_1 \to x_2 \to \dots \to x_k \to \dots$ $\square$

Know $c^T x_1 < c^T x_2 < \dots < c^T x_k < \dots \Rightarrow x_i$ is different from $x_1, x_2, \dots, x_{i-1} \forall i \Rightarrow B_i$ is different from $B_1, B_2, \dots, B_{i-1} \forall i$

Have at most $\binom{n}{m}$ basic $\Rightarrow$ termination.

If t can be 0 $\Rightarrow$ Simplex can cycle (basic maybe repeated) $\Rightarrow$ no termination.

Bland's Rule:

1. Entering variable: pick non-basic entering variable ($c_k > 0$) with the smallest index.
2. Leaving variable: if there are multiple leaving variables, that determinate t, pick the one with the smallest index.

Proposition: Simplex with Bland's Rule terminates.

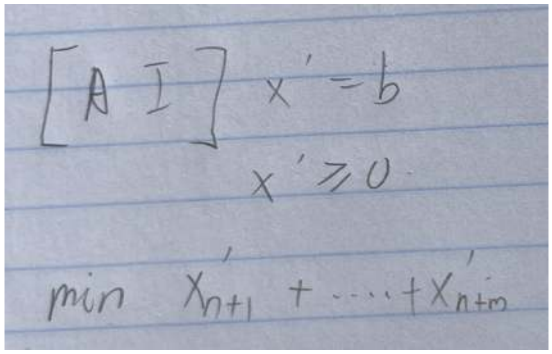
Pf.

Who knows ... :p $\square$

How do we find a feasible solution?

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

$\Rightarrow$ auxiliary problem



$$
\begin{aligned}
\max \quad & \begin{pmatrix} 1 & 2 & -1 & 3 \end{pmatrix} x \\
\text{s.t.} \quad & \begin{pmatrix} 1 & 4 & 2 & 1 \\ -2 & -7 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\
& x \geq 0
\end{aligned}
$$

1. Make sure all rhs are non-negative.
2. Find a feasible solution

   idea: formulate an auxiliary problem that

   (a) is clearly feasible

   (b) is bounded

   (c) lets us determine whether (P) is feasible

   Add one variable for each row of (P) $\Rightarrow x_5, x_6$

We transform the problem into a standard form:

(P):

$$
\begin{aligned}
\Rightarrow \max \quad & \begin{pmatrix} 1 & 2 & -1 & 3 \end{pmatrix} x \\
\text{s.t.} \quad & \begin{pmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
& x \geq 0
\end{aligned}
$$

(P'):

$$\begin{aligned} \Rightarrow \max \quad & -x_5' - x_6' \quad \text{(Turn min into max)} \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 4 & 2 & 1 & 1 & 0 \\ 2 & 7 & 0 & -3 & 0 & 1 \end{pmatrix} x' = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x' \geq 0 \end{aligned}$$

$$x' = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}^T$$

$\Rightarrow$ is feasible $x_5' = 2, x_6' = 1, x_i' = 0$ for $i = 1, 2, 3, 4$

(P') is bounded, all feasible solutions have object value ≥ 0

Note: not in SEF $\Rightarrow$ multiply objective function by -1 (min $\rightarrow$ max)

not in canonical form for $B = \{5, 6\} \Rightarrow y = (1, 1)^T$ [add constraints to objective function to cancel out x_5, x_6]

Running Simplex Method

Opt. Basic $B = \{1, 3\}$

$x' = \begin{pmatrix} \frac{1}{2} & 0 & \frac{3}{4} & 0 & 0 & 0 \end{pmatrix}^T$

objective 0

Feasible solution for original system

Drop the auxiliary variables $x = (\frac{1}{2}, 0, \frac{3}{4}, 0)^T$

Is x basic? We know that $\{1, 3\}$ is basic of aux system

$\Rightarrow \{1, 3\}$ is a basic of (P)

$\Rightarrow$ x base solution for $\{1, 3\}$

One another example

(P)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \begin{pmatrix} 5 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

Finding feasible solution:

1. make sure $b \geq 0$
2. aux problem

$$\begin{aligned} \Rightarrow \min \quad & m_4 + x_5 \equiv \max \quad -m_4 - x_5 \\ \text{s.t.} \quad & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

Solve aux problem using simplex: $B = \{3, 5\}$

$x = (0, 0, 1, 0, 2)^T \Rightarrow$ aux LP has value $z > 0$

Claim: (P) is infeasible

Proof. To Suppose for contradiction that (P) is feasible

$\Rightarrow x : Ax = b, x \geq 0$

$\Rightarrow x' = (x, 0, 0)^T$ feasible for aux LP, objective value 0.

Contradiction that aux LP has optimal value $z > 0$ □

viii. 2-Phase Method

(P)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Phase 1: Find a feasible solution to (P) or determine that (P) is infeasible

Phase 2: Given basic feasible solution from Phase 1, run simplex method on (P)

Note: Simplex with bland's rule terminates in finite number of steps

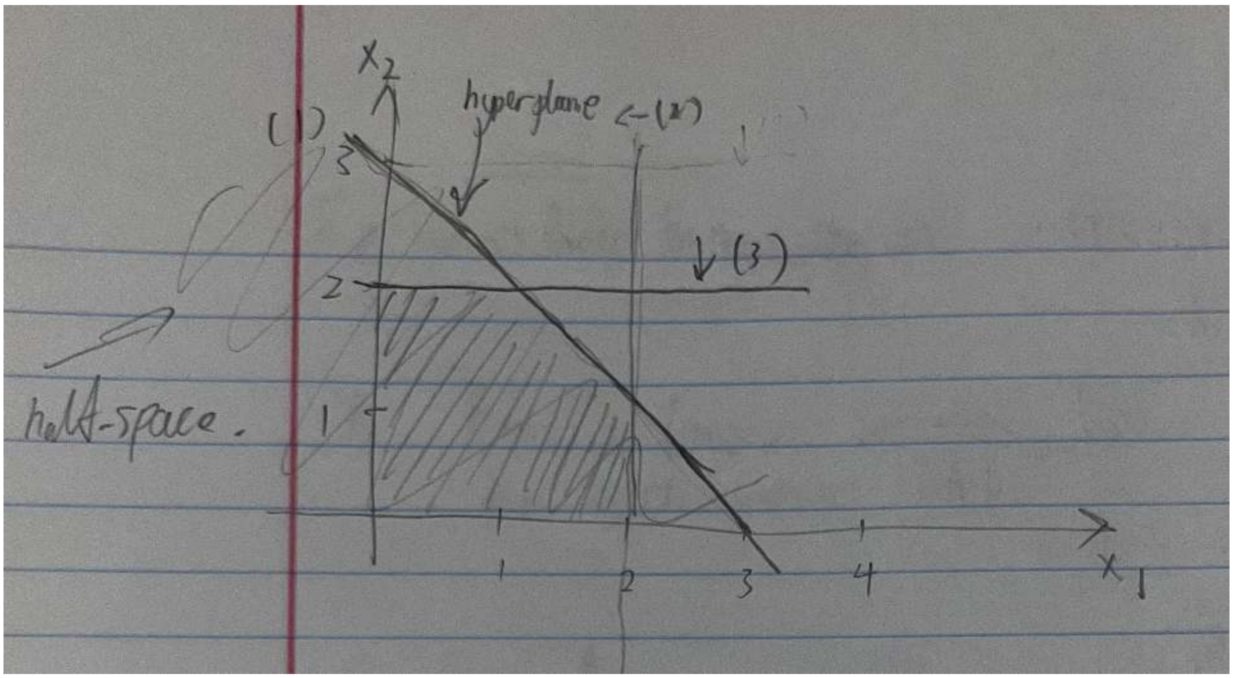
Run 2-Phase method with Simplex / Bland

2-Phase determinate that (P) is either infeasible, unbounded or it has an optimum solution fundamental theorem of LP.

ix. Geometry of LP

(P)

$$\begin{array}{ll} \max & (1,2)x \\ \text{s.t.} & \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \end{array}$$



Def :
$P \subseteq \mathbb{R}^n$ is called a polyhedron if it can be written as $\{x : F(x) \le b\}$ **Note:** the feasible region of any LP

is a polyhedron

$$\begin{array}{ll} \max & (3,2,1)x \\ \text{s.t.} & \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ & x \ge 0 \end{array}$$

Feasible region is a polyhedron $\begin{pmatrix} 2 & 0 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{aligned} & a^T x = b \quad a \subseteq \mathbb{R}^n, b \in \mathbb{R} \\ \Leftrightarrow\ & a^T x \leq b \\ & -a^T x \geq -b \end{aligned}$$

Understand polyhedra

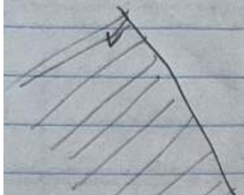
<u>Def</u> :
Let $a \subseteq \mathbb{R}^n$ vector, $\beta \in \mathbb{R}$

1) $\{x : a^T x = \beta\}$ is a hyperplane
2) $\{x : a^T x \leq \beta\}$ is a half-space

$\Rightarrow$ A polyhedron is the intersection of a finite set of half-spaces

x. Geometry of Feasible Area

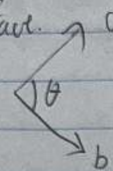
$H = \{x : a^T x \leq \beta\}$ half-space, $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$



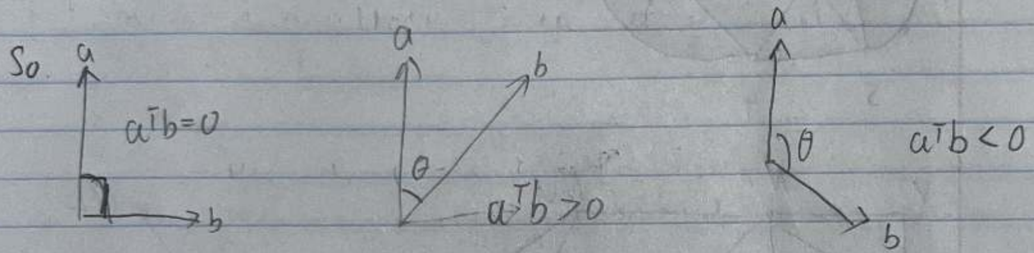
hyperplane: $\{x : a^T x = \beta\}$

Recall: $a^T b = \|a\| \|b\| \cdot \cos\theta$ (norm)

Fact.



So.



$H = \{x : a^T x = 0\} \rightarrow H$ contains all x that are orthogonal to a.

Def: Call S' a translate of $S' \subseteq \mathbb{R}^n$ if $S' = \{x+p, x \in S\}$ for some fixed $p \in \mathbb{R}^n$

Proposition: Let $a \neq 0$, $a \in \mathbb{R}^n$, $H = \{x : a^T x = \beta\}$ is a translate of $H^0 = \{x : a^T x = 0\}$

Pf: choose some $p \in H$ (exists b/c $a \neq 0$)

$x \in H^0 \Leftrightarrow x + p \in H$

$x \in H^0 \Leftrightarrow a^T x = 0 \Leftrightarrow a^T x + a^T p = \beta$

$\Leftrightarrow a^T(x+p) = \beta$

$\Leftrightarrow x + p \in H$

Similar: half space $\{x : a^T x \leq \beta\}$ is a translate of $\{x : a^T x \leq 0\}$

$H = \{x : a^T x = \beta\}$, $\beta \neq 0$ — H is not a vector space

$H_0 = \{x : a^T x = 0\}$ — vector space.

Def: The dimension of H (half space / hyperplane) is the dimension of its translate H^0

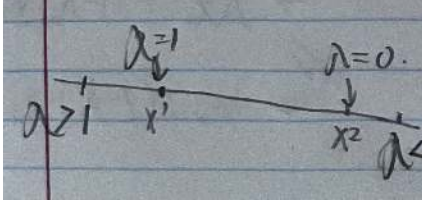
$\Rightarrow \dim(H) = \dim(H^0) = n - \text{rank}(a) = n-1$

(H: hyperplane, $a \in \mathbb{R}^n$)

$P = \{x : Ax \leq b\}$ polyhedron

ex.

Def: $x^1, x^2 \in \mathbb{R}^n$, the line through x^1, x^2 is given by



$L = \{x = \lambda x_1 + (1-\lambda) x_2,\ \lambda \in \mathbb{R}\}$

The line segment between x_1, x_2

$$S = \{ x = \lambda x^1 + (1-\lambda) x^2 ,\ \lambda \in [0,1] \}.$$

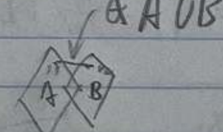
Def: A set $S \subseteq \mathbb{R}^n$ is <u>convex</u> if $\forall x_1, x_2 \in S$ the line segment between x_1, x_2 is constraint in S.

If we want to show that S', convex it surface to show that $\lambda x_1 + (1-\lambda)x_2 \in S$ $\forall x_1, x_2 \in S$, $\lambda \in [0,1]$.

Similar show S' is not convex amount to finding $x_1, x_2 \in S'$, $\lambda' \in [0,1]$, $\lambda x_1 + (1-\lambda) x_2 \notin S'$

Q: $A, B \subseteq \mathbb{R}^n$ convex set

1) $A \cup B$ is it convex ? No



2) $A \cap B$ is it convex ? Yes

Prop. $A, B \subseteq \mathbb{R}^n$ convex $\Rightarrow A \cap B$ convex

p.f. $x_1, x_2 \in A \cap B$, $\lambda \in [0,1]$

need to show $\lambda x_1 + (1-\lambda) x_2 \in A \cap B$.

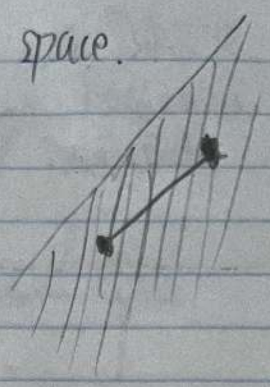
$S' \in \{A, B\}$, $x_1 x_2 \in S'$ and S convex

$\hookrightarrow \lambda x_1 + (1-\lambda) x_2 \in S'$

$\hookrightarrow \lambda x_1 + (1-\lambda) x_2 \in A \cap B$

Prop: $H = \{x : a^T x \le \beta\}$ half space.

H is convex



Pf. $\forall x_1, x_2 \in H,\ \lambda \in [0,1]$

$a^T x^i \le \beta \quad \forall i \in \{1,2\}$

to show $\lambda x_1 + (1-\lambda) x_2 \in H$, $a^T(\lambda x_1 + (1-\lambda) x_2) \in \beta$

$$a^T(\lambda x_1 + (1-\lambda) x_2) = \lambda \underbrace{a^T x_1}_{\le \beta} + (1-\lambda) \underbrace{a^T x_2}_{\le \beta}.$$

$$\le \lambda \beta + (1-\lambda)\beta = \beta \quad \blacksquare$$

hyperplane are also convex !

$P = \{x : Ax \le b\}$.

$A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ $a_i \in \mathbb{R}^n$

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$

$= \{x : a_1^T x \le b_1\} \cap$

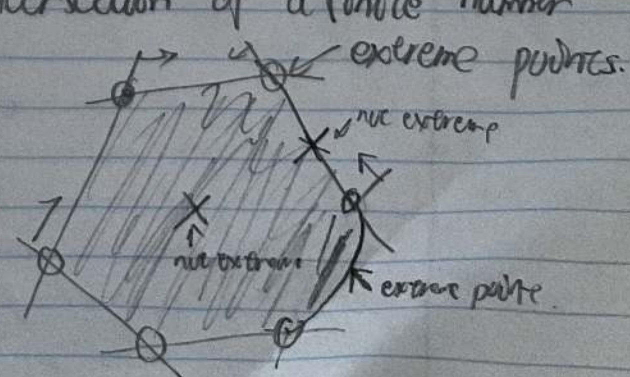
$\{x : a_2^T x \le b_2\} \cap$

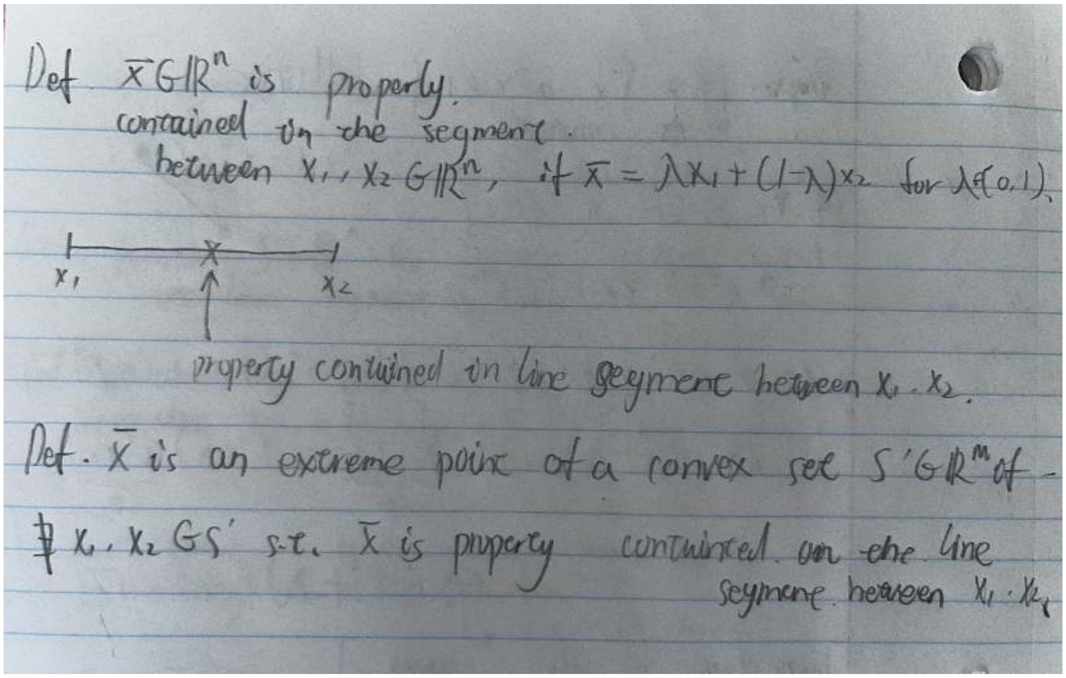
$\vdots$

$\{x : a_m x \le b_m\}$

$\Rightarrow$ A polyhedron is the intersection of a finite number of half space.

$\Rightarrow$ Polyhedron are convex





Certificate of optimal, $y \in \mathbb{R}^m$, $y^T A \geq c^T$ for all $c \in \mathbb{R}^n$.

$$\Rightarrow c^T x \underbrace{\leq}_{x \geq 0} y^T A x \underbrace{\leq}_{x fea} y^T b$$

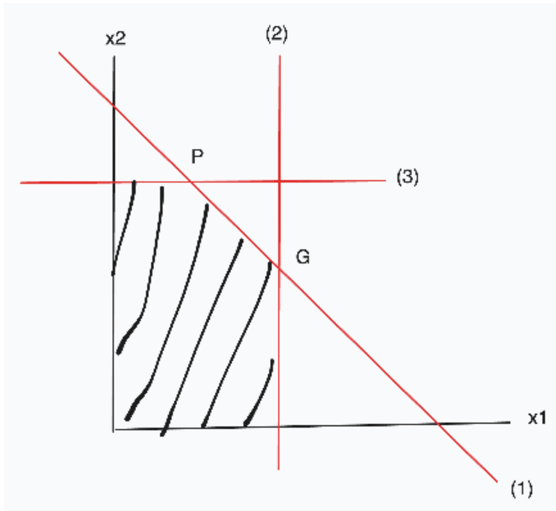
If $c^T x = y^T b \Rightarrow$ x optimal

Geometry $S \in \mathbb{R}^n$,

$x \in S'$ is an extreme point if $\nexists x^1, x^2 \in S'$, $x = \lambda x^1 + (1-\lambda)x^2$, $0 < \lambda < 1$

Q: $H = \{x : \underbrace{a^T}_{\mathbb{R}^n} x \leq \underbrace{\beta}_{\mathbb{R}}\}$, Does H have extreme points? No!

Example:



$P = (1,2)^T, G = (2,1)^T$ extreme points.

At P, (1) and (3) hold with equality. We say (1) and (3) are tight at P.

Notice: A_1 and A_3 are linearly independent, Similarly A_1 and A_2 are linearly independent.

Def :

$\bar{x} \in P$, Let $\bar{A}x \leq \bar{b}$ be the set of inequalities of $Ax \leq b$ that hold with equality at $\bar{x}$.
The sub-system of $Ax \leq b$ of tight constraints at $\bar{x}$.

Example:

$$\bar{x} = (1,2)^T,\ \bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \bar{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Theorem :

$\bar{x}$ is an extreme point of P if and only if $rank(\bar{A}) = n$. $(A \in \mathbb{R}^{m \times n})$
($\bar{A}x \leq \bar{b}$ is a sub-system of tight constraints at $\bar{x}$)

Pf.

"$\Leftarrow$" Let $\bar{x} \in P$ and equality/tight sub-system, such that $\bar{A}x \leq \bar{b} \rightarrow \bar{A}\bar{x} = \bar{b}$
Let's suppose $\bar{x}$ for constraints that $\bar{x}$ is not an extreme point $\Rightarrow \exists x^1, x^2 \in P$, s.t. $\bar{x} = \lambda x^1 + (1-\lambda)x^2$, $0 < \lambda < 1$

$$\begin{aligned} \bar{b} = \bar{A}\bar{x} &= \bar{A}(\lambda x^1 + (1-\lambda)x^2) \\ &= \lambda \bar{A}x^1 + (1-\lambda)\bar{A}x^2 \\ &\leq \lambda \bar{b} + (1-\lambda)\bar{b} \\ &= \bar{b} \end{aligned}$$

$\Rightarrow \bar{A}x^1 = \bar{A}x^2 = \bar{b}$
$\Rightarrow \bar{A}$ is linearly dependent, which is a contradiction.

"$\Rightarrow$" Prove contrapositive: $rank(\bar{A}) < n \Rightarrow \bar{x}$ is not an extreme point.
$rank(\bar{A}) < n \Rightarrow \exists d \in null(\bar{A})$, $d \neq 0$
$\Rightarrow x^1 = \bar{x} - \epsilon d$, $x^2 = \bar{x} + \epsilon d$, $0 < \epsilon < \infty$
$\Rightarrow \bar{x} = \frac{1}{2}(x^1 + x^2)$, $\bar{x}$ is not an extreme point.

Consider inequalities $a^T x \leq \beta$, not in $\bar{A}x \leq \bar{b}$, $\Rightarrow a^T\bar{x} < \beta$
$a^T x^1 = a^T\bar{x} - \epsilon a^T d < \beta$, for some small enough $\epsilon \neq 0$
Choose $\epsilon \neq 0$ s.t. all inequalities not in $\bar{A}x \leq \bar{b}$ hold for both x^1 and x^2. $\square$

Example:

$$\bar{P} = \left\{ x \geq 0, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$$

Write in polyhedron form:

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ -2 \\ 4 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Observe $\bar{x} = (2, 4, 0)^T$ is a basic feasible solution of $\bar{p}$ and $\bar{x}$ is an extreme point of $\bar{P}$.

Theorem :

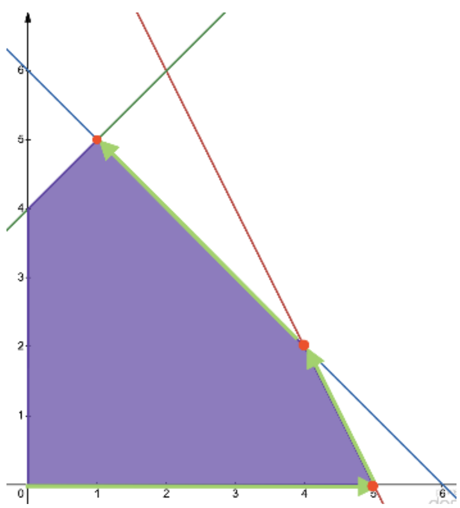
$P = \{x \geq 0, Ax = b\}$ is LP is SEF, Then the following are equivalent:

1. $\bar{x}$ is a basic feasible solution of P
2. $\bar{x}$ is an extreme point of its feasible region

Simplex Method Shows on the Board

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix}$$

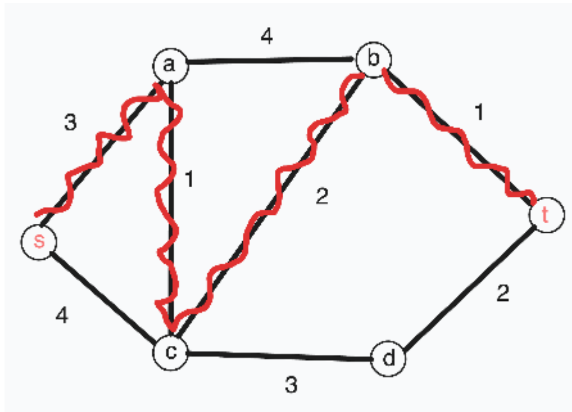


$$\bar{P} = \left\{ x \geq 0 : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Simplex produces the following basic solution:

$$\begin{pmatrix} 0 \\ 0 \\ 10 \\ 6 \\ 4 \end{pmatrix} \to \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 9 \end{pmatrix} \to \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 6 \end{pmatrix} \to \begin{pmatrix} 1 \\ 5 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

xi. Shortest Paths



- $G = (V, E)$, $s, t \in V$,
- edge $uv \in E$ has length C_{uv}
- Find s.t. path
  - $p = u0u1, u1u2, \ldots, u_{i-1}u_i \; u_iu_{i+1} \in E$
  - $u_i \neq u_j$ for $i \neq j$
  - $u_0 = s$, $u_{n+1} = t$

Length of p:

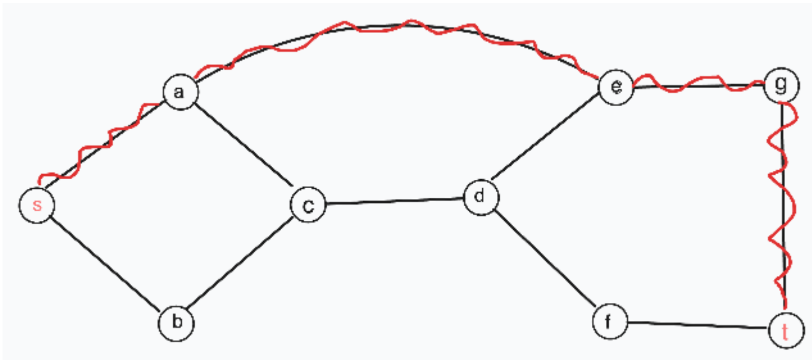
$$c(p) = \sum_{e \in p} c_e$$
$$= 7$$

By inspection, p is the shortest st-path $\rightarrow$ no shorter path between s and t exists.

Q:

- Given P, can we prove that it is the shortest path? (Certification)
- Can we find the shortest path efficiently?

Simple case: cordiality special case $c_e = 1, \forall e \in E$

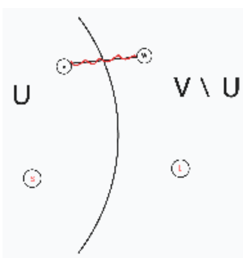


Shortest path: $p = sa, ae, eg, gt \;\; c(p) = 4$

<u>Def</u> :
$u \subseteq v$, define cut induced by u

$$\delta(u) = \{vw : v \in u, w \notin u\}$$

Call $\delta(u)$ an $s, t - cut$ $(s.t. \in V)$ if $s \in u, t \notin u$



Observe:

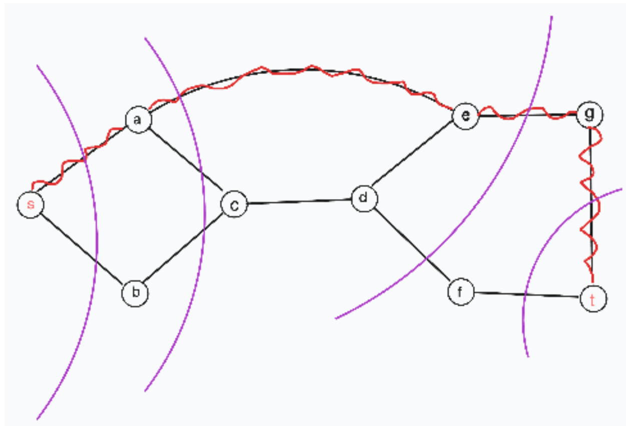
- P s.t-path, $\delta(u)$, s, t-cut, then p contains at least one edge from $\delta(u)$

Back to cordiality example:

<u>Def</u> :
$u_1 = \{s\}, u_2 = \{s, a, b\}, u_3 = \{s, a, b, c, d, e\}, u_4 = V \backslash \{t\}$

$\Rightarrow \delta(u_i) \cap \delta(u_j) = \emptyset$ for $i \neq j$

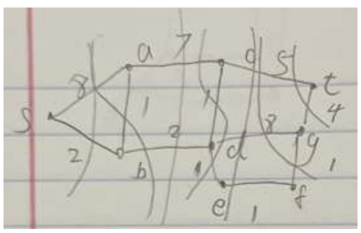
Example:



If p is an s, t-path, then

$$p \cap \delta(u_i) \neq \emptyset \quad \forall i = 1, \dots, 4, e_i \neq e_j, \forall i \neq j$$

$\Rightarrow$ st-path p must have ≥ 5 edges. Path p we looked at earlier is the shortest path.



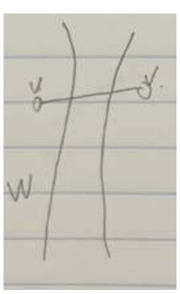
$P = sb, bd, dc, ct \quad c(P) = 10$

Width assignment:

$$y = \{yu : \delta(u)s, t - cut\}$$

y feasible if

- $y \geq 0$
- $\sum_{u \in U(e)} y_u \leq c_e \quad \forall e \in E$



$u(u,v) = \{W \subseteq V, u \in W, s \in W, t \notin W, v \notin W\}$

We showed: y feasible width assignment, p s, t-path. $c(P) \geq \sum_{u \in U(P)} y_u$

If $\sum_{e \in E} y_u = c(P) \Rightarrow p$ is the shortest path.

u	y_u
s	2
$\{s, b\}$	1
$\{s, a, b\}$	1
$\{s, a, b, d\}$	1
$\{s, a, b, c, d, e\}$	1
$\{s, a, b, c, d, e, f\}$	1
$\{t\}$	3

$\sum_u y_u = 10$ $y_u = 0$ if u does not appear in the table.

Back to LP.

$$\begin{aligned} \min \quad & (2,3)x \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

$x^1 = (0, 20) \rightarrow c^T x = 60$ Not optimal.

$x^2 = (5, 13) \rightarrow c^T x = 49$

Find a puof that x^2 is optimal.

Suppose x is feasible for (P)

x satisfies

$$\begin{aligned} (2,1)x &\geq 20 \\ +(1,1)x &\geq 18 \\ +(-1,1)x &\geq 8(2,3)x \qquad \geq 46 \end{aligned}$$

Any feasible x must have

$$z(x) = (2,3)x \geq 46$$

For any feasible x, objective function value $z(x)$ of x must be ≥ 46.

Can we find a better lower bound on $z(x)$?

The inequality

$$y^T \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} x \geq y^T \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix}$$

x satisfies $Ax \geq b$ then x satisfies $y^T Ax \geq y^T b$

Choosing $y = (1,1)^T$

$\Rightarrow y^T Ax \geq y^T b$

$\Rightarrow (2,3)x \geq 46$

Another example,

$y = (0,2,1)^T$

$$\begin{aligned} & y^T Ax \geq y^T b \\ \Rightarrow & (1,3)x \geq 44 \\ \Rightarrow & (1,3)x - 44 \geq 0 \end{aligned}$$

For all feasible x

$$\begin{aligned} z(x) &= (2,3)x \\ & (2,3)x + 44 - (1,3)x \\ &= (1,0)x + 44 \geq 44 \end{aligned}$$

Lower Bound argument hinges on

1. $y \geq 0$
2. $y^T A \leq (2,3) = c^T$
3. want: $y^T b$ large

Does the LP have an optimal solution?

- The LP is feasible: $y = 0$
- Bounded: $z(x) \leq$ the objective value of any feasible solution to original LP $\Rightarrow z(x) \geq 49$

$\Rightarrow$ has an optimal solution

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

x feasible for (P), $y \geq 0$

$$\begin{aligned} &\Rightarrow y^T A x \geq y^T b \\ &\Leftrightarrow y^T b - y^T A x \leq 0 \end{aligned}$$

$\forall$ feasible x for (P):

$$z(x) = c^T x \geq c^T x - y^T A x + y^T b = (c^T - y^T A)x + y^T b$$

If $c^T - y^T A \geq 0$ and $y^T b$ is large, then $z(x) \geq y^T b$

Find the $y \geq 0$ that gives largest lower bound on optimal value of (P)

(D) is the LP Dual of (P)

(P) is called the LP Primal

(D) is

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

Theorem Weak Duality Theorem:
If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D), then $c^T \bar{x} \geq b^T \bar{y}$

Pf.

$$b^T \bar{y} \leq (A\bar{x})^T \bar{y} = \bar{x}^T (A^T \bar{y}) \leq c^T \bar{x}$$

□

xii. Weak Duality

(P)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

(Primal)

(D)

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

(Dual)

Theorem Weak duality:
$\bar{x}$,$\bar{y}$ feasible for (P) and (D)

$$c^T \bar{x} \geq b^T \bar{y}$$

Back to the shortest Paths(SP) $G = (V, E)$, $c_e \geq 0 \forall e \in E, s, t \in V$.

MIP(Mix Integer Program) for SP

(SP-IP) min $\quad \sum_{e \in E} c_e x_e$

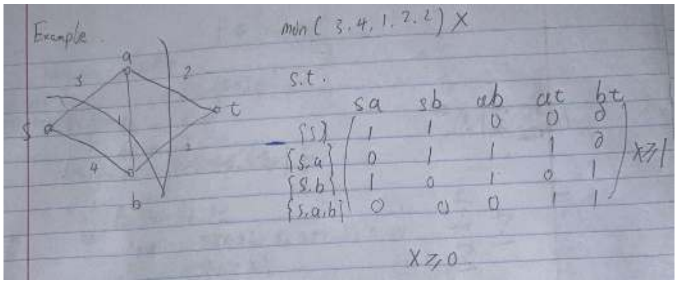
(SP-P) s.t. $\quad \sum_{e \in \delta(w)} x_e \geq 1 \quad \forall w \in U$

$x \geq 0$ ~~(x integer)~~ LP relaxation of SP-IP

$U = \{w \subseteq V, s \in W, t \notin W\}$

$\delta(w) = \{uv \in E, u \in W, v \notin W\}$

Example:



$P = sa, at \quad s, t - path$

Edge-incidence vector of P: $x_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, x_{p,a} = \begin{cases} 1 & \text{if } a \in P \\ 0 & \text{otherwise} \end{cases}$

Note: p is an s, t-path $\Rightarrow x_p$ is feasible for shortest path IP (SP-IP).
Compute the dual of (SP-P).

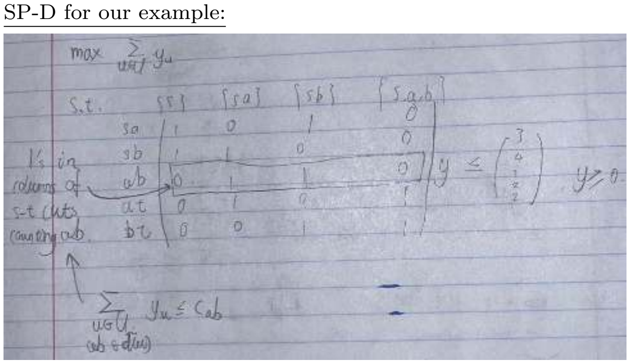
(SP-P) (Weak Duality $c^T x \geq \mathbb{1}^T \bar{y}$ for feasible $\bar{x}$ and $\bar{y}$)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq \mathbb{1} \\ & x \geq 0 \end{aligned}$$

(SP-D)

$$\begin{aligned} \max \quad & \mathbb{1}^T y \\ \text{s.t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

SP-D for our example:



y feasible for (SP-D) $\Rightarrow y \geq 0$ and y feasible with assignment.

1. General argument

(SP-P)

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(w)} x_e \geq 1 \quad \forall u \in U \\ & x \geq 0 \end{aligned}$$

$\{W \subseteq V, s \in W, t \notin W\}$

SP-D

$$\begin{aligned} \max \quad & \sum_{w \in U} y_w \\ \text{s.t.} \quad & \sum_{w: uv \in \delta(w)} y_w \leq c_{uv} \quad \forall uv \in E \\ & y \geq 0 \end{aligned}$$

Note: y feasible for (SP-D) $\Rightarrow$ y feasible width

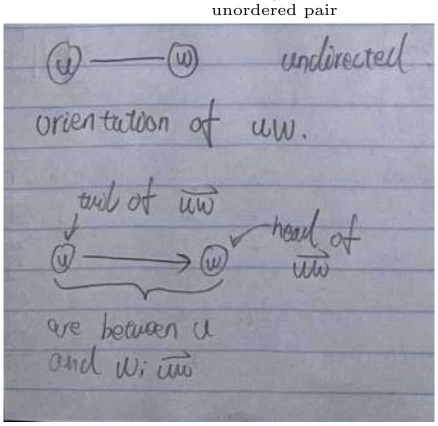
Weak duality $\Rightarrow$

- p is a st-path, x_p is feasible for (SP-P)
- if y is a feasible width assignment $\Rightarrow$ y is feasible for (SP-D)

$c(p) = c^T x_p \geq \mathbb{1}^T y = \sum_{u \in U} y_u$

xiii. Algorithm of SP

Lingo: $G = (V, E)$, $\underbrace{uw}_{\text{unordered pair}} \in E$,



Path: $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$

directed path $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{k-1}v_k}$

directed s, t-path, if also $v_1 = s, v_k = t$

(Note: $\overrightarrow{v_iv_{i+1}}$ are arcs in our graph, $v_i \neq v_j \forall i \neq j$)

1. SP instance

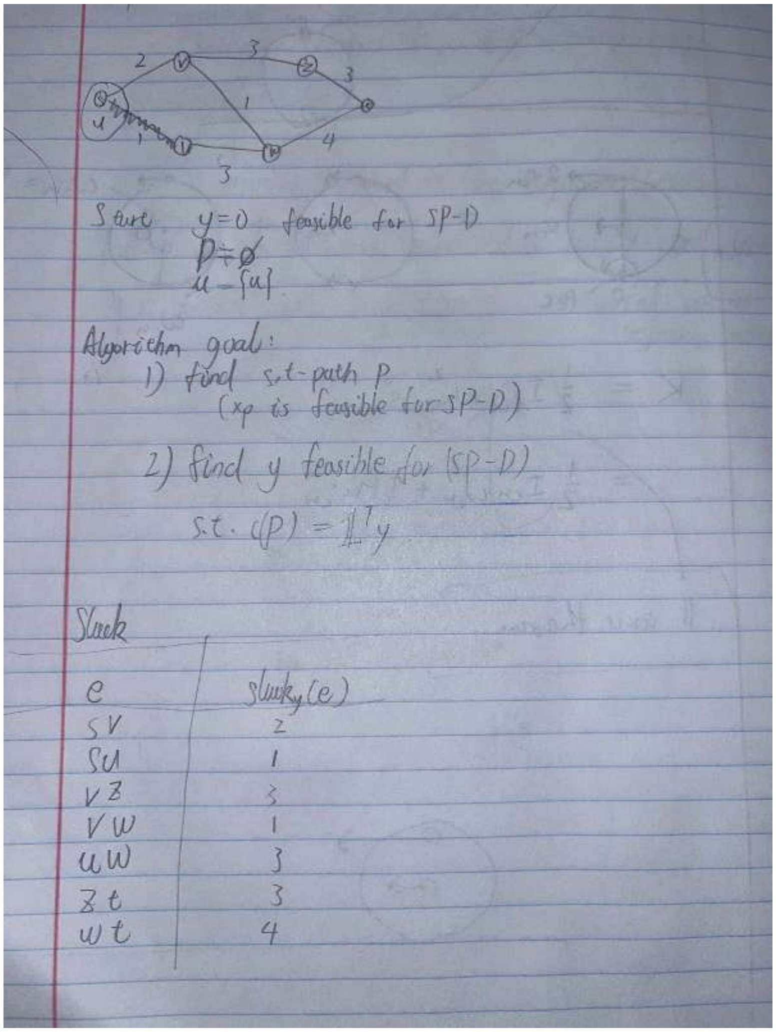
$G = (U, E), c_e \geq 0 \forall e \in E, s, t \in V$

The slack ($slack_y(e)$) of an edge e for duality is the gap between the left-hand side and the right-hand side of the dual constraint corresponding to e.

$$\begin{aligned} \max \quad & \sum_{u \in U} y_u \\ \text{s.t.} \quad & \sum_{u, e \in \delta(u)} y_u \leq \quad c_e \forall e \in E \\ & y \geq 0 \end{aligned}$$

$$slack_y(e) = c_e - \sum_{u:e\in\delta(u)} y_u$$

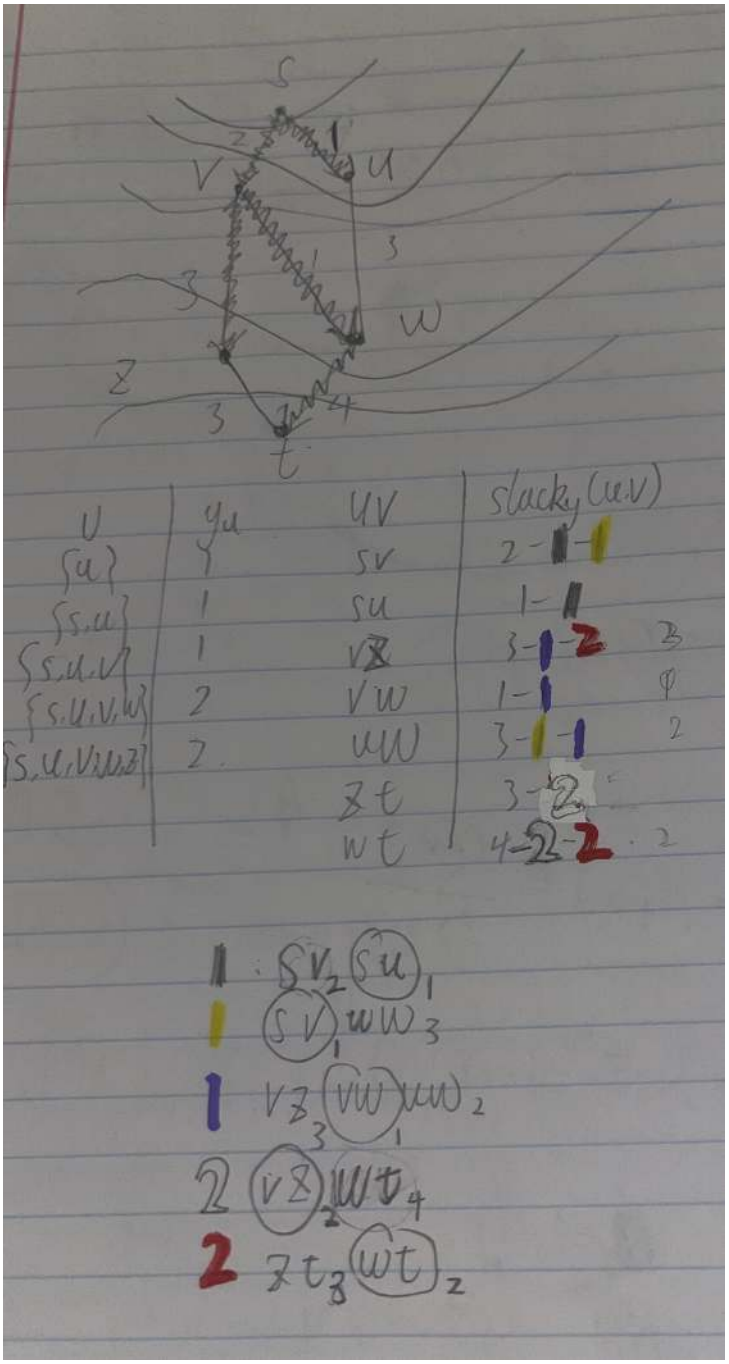
Example:



Iteration of SP algorithm:

- Find $e \in \delta u$ with minimum $slack_y(e)$
- $y_u = slack_y(e)$

xiv. Shortest Path Algorithm



Input: $G = (V, E)$, $c_e \geq 0$, $\forall e \in E$, $s, t \in V$.

1. $y = 0$, $u = \{s\}$

2. while $t \notin u$
3. Find $e = \delta(u)$ with min $slack_y(e) \Rightarrow sa$
4. $y_u = slack_y(e) \Rightarrow y_u = 1$
5. $u = u \cup \{b\}$
6. turn e into arc $\overrightarrow{ab}$

$u : \{w \subseteq V : s \in w, t \notin w\}, u_e = \{u \in U : e \in \delta(u)\}$

(D)

$$\begin{aligned} \max \quad & \sum_{u \in U} y_u \\ \text{s.t.} \quad & \sum_{u \in U_e} y_u \leq c_e, \quad \forall e \in E \\ & y \geq 0 \end{aligned}$$

uv	y_u $slack(uv)$
sv	2
su	1
vz	3
vw	1
uw	3
zt	3
wt	4

$$slack(e) = c_e - \sum_{u \in U_e} y_u$$

Lingo:

- Call $uv \in E$ an equality edge for dual y if slack $(e) = 0$
- Call $\delta(u)$ active of $y_u > 0$

Proposition: Let y be a feasible dual solution, P an $s, t - path$.

Then P is the shortest path if

(i) all edges of pare equality edges

(ii) all active cuts $\delta(u)$ have exactly same $p - edge$.

$\underline{\underline{Pf.}}$

p,y satisfy (i) and (ii).

$$c(p) = \sum_{e \in p} c_e \underbrace{=}_{\text{all edges on pare equality}} \sum_{e \in p} \sum_{u \in u_e} y_u$$
$$= \sum_{u \in U} y_u |\delta(u) \cap p| = \mathbb{1} y$$

$u : \{w \subseteq v : s \in U, t \notin U\}$

$u_e = \{w \in U : e \in \delta(w)\}$

p compounds to an optical feasible solution x_p

$$c(p) = c^T x_p = \mathbb{1} y$$

$\Rightarrow$ P is the optimal path. □

(D)

$$\begin{aligned} \max \quad & \sum_{u \in U} y_u \\ \text{s.t.} \quad & \sum_{u \in U_e} y_u \leq c_e \\ & y \geq 0 \end{aligned}$$

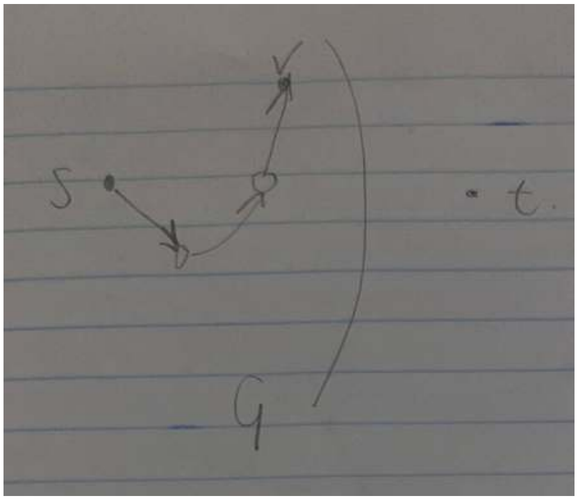
Proposition: The shortest path Algorithm maintains

(I) y is a feasible dual solution

(II) arcs are equality edges (i.e. have Slack 0)

(III) no active cut has an entering edge. $\nexists \overrightarrow{wv}, v \in \delta(u), w \notin \delta(u), y_u > 0$

(IV) $\forall u \in U, \exists$ directed s, u-path



(V) arcs have both ends in u

$\Rightarrow$ at termination $\exists$ s.t - directed path

Suppose this is true and (I) - (V) hold when the Algorithm terminates.

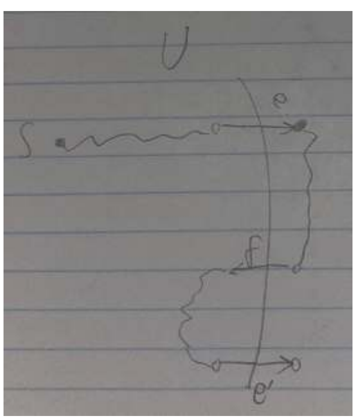
1. $\exists$ s,t-path directed path p (IV)
2. y is dual feasible (I)
3. all arcs on P are equality edges (II)

to show: $\delta(u)$ active $|p \cap \delta(u)| = 1$

Suppose: $\exists u, y_u > 0$ and $|p \cap \delta(u)| \geq 2$

Let e, e' be the fint two arcs on p. s.t. their tail is in U and head is not in U.

$\Rightarrow \exists$ arc f on p that has tail outside u, and head inside u.



(III) no active cut has an entering arc $\Rightarrow f$ would violate (III).

Note: missing invariant I - V are maintained inductively by the Algorithm.

(SP)

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(w)} x_e \geq 1 \quad \forall w \in U[yw] \\ & x \geq 0 \end{aligned}$$

(D)

$$\begin{aligned} \max \quad & \sum_{u \in U} y_u \\ \text{s.t.} \quad & \sum_{u \in U_e} y_u \leq c_e \quad \forall e \in E \\ & y \geq 0 \end{aligned}$$

Theorem :
$\bar{x}$ feasible for (SP), $\bar{y}$ feasible for (D), then

$$\Rightarrow c^T \bar{x} \leq \mathbb{1}^T \bar{y}$$

Note:

(1) Primal constraint $\equiv$ Dual variable

(2) Primal variable $\equiv$ Dual constraint

(3) Primal non-negatively $\equiv$ Dual "$\leq$" constraint

(4) Primal "$\geq$" constraint $\equiv$ Dual non-negatively

General

(P_{max})

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \quad ? \quad b \\ & x \quad ? \quad 0 \end{aligned}$$

D_{min}

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \quad ? \quad c \\ & y \quad ? \quad 0 \end{aligned}$$

P_{max}	D_{min}
(constraint)	
'$\leq$' constraint	≥ 0 constraint
'$=$' constraint	free variable
'$\geq$' constraint	≤ 0 constraint
(Non-negatively)	
'≥ 0' variable	'$\geq$' constraint
free variable	'$=$' constraint
'≤ 0' variable	'$\leq$' constraint

Example:

(P)

$$\begin{aligned} \max \quad & (2,4,1)x \\ \text{s.t.} \quad & \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & 5 \end{pmatrix} x \begin{pmatrix} = \\ \leq \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \\ & x_1 \geq 0, x_2 \geq 0, x_3 \text{ free} \end{aligned}$$

(D)

$$\begin{aligned} \min \quad & (3,5)y \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 2 & 5 \end{pmatrix} y \begin{pmatrix} \geq \\ \geq \\ = \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \\ & y \geq 0 \end{aligned}$$

Note:

1. P_{max} side of table
2. dual non-negatively
   y_1 free, $y_2 \geq 0$
3. dual constraint

   constraint signs: $\begin{pmatrix} \geq \\ \geq \\ = \end{pmatrix}$

Example:

(P)

$$\begin{aligned} \min \quad & d^T y \\ \text{s.t.} \quad & w^T y \geq e \\ & y \geq 0 \end{aligned}$$

Compute dual of (P)

1. (P) is a min LP $\Rightarrow$ right side of table

   (D)

$$\begin{array}{rl} \max & e^T x \\ \text{s.t.} & wx \quad ? \quad d \\ & x \quad ? \quad 0 \end{array}$$

   - d $\equiv$ b
   - $w^T \equiv A^T$
   - e $\equiv$ c

2. dual non-negatively

   (P) has '$\Rightarrow$' constraint $\Rightarrow$ '≥ 0' variable

3. dual constraints

   (P) '≥ 0' variables $\Rightarrow$ '$\leq$' constraints

Theorem :

Let P_{max} and P_{min} be primal LP and its dual.
If $\bar{x}$ is feasible for P_{max} and $\bar{y}$ is feasible for P_{min},
Then $c^T \bar{x} \leq b^T \bar{y}$

(P_{max})

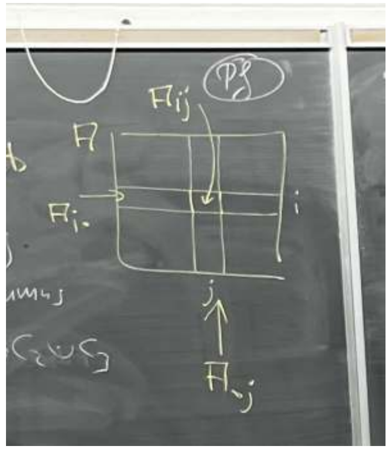
$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & Ax \quad ? \quad b \\ & x \quad ? \quad 0 \end{array}$$

(D_{min})

$$\begin{array}{rl} \min & b^T y \\ \text{s.t.} & A^T y \quad ? \quad c \\ & y \quad ? \quad 0 \end{array}$$

General LP

$$\begin{array}{rl} \max & c^T x \\ \text{s.t.} & A_i x \leq b_i \quad i \in R_1 \\ & A_i x \geq b_i \quad i \in R_2 \\ & A_i x = b_i \quad i \in R_3 \\ & x_j \geq 0 \quad j \in C_1 \\ & x_j \leq 0 \quad j \in C_2 \\ & x_j \text{ free} \quad j \in C_3 \\ & C = \{1 \ldots n\} \\ & C = C_1 \cup C_2 \cup C_3 \end{array}$$



(Dual)

$$\begin{aligned}
\min \quad & b^T y \quad b = (b_i)_{i \in R_1} \\
\text{s.t.} \quad & A_j^T y \ge c_j \quad j \in R_1 \\
& A_j^T y \le c_j \quad j \in R_2 \\
& A_j^T y = c_j \quad j \in R_3 \\
& y_i \ge 0 \quad i \in C_1 \\
& y_i \le 0 \quad i \in C_2 \\
& y_i \text{ free} \quad i \in C_3 \\
& y_i \ge 0 \quad i \in R_1 \\
& y_i \le 0 \quad i \in R_2 \\
& y_i \text{ free} \quad i \in R_3
\end{aligned}$$

Remark:

(P)

$$\begin{aligned}
\max \quad & c^T x \\
\text{s.t.} \quad & Ax + s = b \\
& s_i \ge 0 \quad \forall i \in R_1, \quad x_j \ge 0 \quad \forall j \in C_1 \\
& s_i \le 0 \quad \forall i \in R_2, \quad x_j \le 0 \quad \forall j \in C_2 \\
& s_i = 0 \quad \forall i \in R_3, \quad x_j \text{ free} \quad \forall j \in C_3
\end{aligned}$$

(D)

$$\begin{aligned}
\min \quad & b^T y \\
\text{s.t.} \quad & A^T y + w = c \\
& w_i \ge 0 \quad \forall i \in C_1, \quad y_j \ge 0 \quad \forall j \in R_1 \\
& w_i \le 0 \quad \forall i \in C_2, \quad y_j \le 0 \quad \forall j \in R_2 \\
& w_i = 0 \quad \forall i \in C_3, \quad y_j \text{ free} \quad \forall j \in R_3
\end{aligned}$$

Suppose $\bar{x}$, $\bar{y}$ are feasible (P), (D).
Let: $\bar{s} = b - A\bar{x}$, $\bar{w} = c - A^T\bar{y}$

$\Rightarrow (\bar{x}, \bar{s}), (\bar{y}, \bar{w})$ are feasible for (P), (D)
$\rightarrow \bar{y}^T b = \bar{y}^T A\bar{x} + \bar{y}^T \bar{s} = (c - \bar{w})^T \bar{x} + \bar{y}^T \bar{s} = c^T \bar{x} - \bar{w}^T \bar{x} + \bar{y}^T \bar{s} \geq c^T \bar{x}, \quad (\bar{w} \geq 0, \bar{s} \geq 0)$

(P_{max})

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \quad ? \quad b \\ & x \quad ? \quad 0 \end{aligned}$$

(P_{min})

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A^T y \quad ? \quad b \\ & y \quad ? \quad 0 \end{aligned}$$

1. Strong Duality

If P_{max} has an optimal solution $\bar{x}$, then P_{min} has an optimal solution $\bar{y}$, and $c^T \bar{x} = b^T \bar{y}$.

(P_{max})

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

(P_{min})

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A^T y \geq b \end{aligned}$$

Suppose $\bar{x}$ is an optimal solution for P_{max}.

$\rightarrow$ Know that 2-phase Simplex terminates with an optimal basic solution.

2nd phase of 2-phases terminates with basis B Canonical form.

$$
\begin{aligned}
\max \quad & \bar{c}^T\bar{x} + \bar{y}^T b \\
\text{s.t.} \quad & x_B + A_B^{-1}(A_N x_N) = A_B^{-1} b \\
& x_B, x_N \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\bar{y} &= A_B^{-T}\bar{c}_B \\
c^T &= c^T - \bar{y}^T A
\end{aligned}
$$

(P') and P_{max} are equivalent.

Let $\bar{x}$ be optimal solution for B.

$c^T\bar{x} = \bar{y}^T b + \bar{c}^T\bar{x} = y^T b + \underbrace{\bar{c}_N^T\bar{x}_N}_{0} + \underbrace{\bar{c}_B^T\bar{x}_B}_{0} = \bar{y}^T b$

Claim: $\bar{y}$ is feasible for $P_{min} = \bar{y}^T b$

Pf.

Optimality

$\bar{c} = c - A^T\bar{y} \leq 0$

$\Rightarrow A^T\bar{y} \geq c$ □

(P_{min}) and (P_{max}) feasible

$\Rightarrow$ (weak duality) (P_{min}) and (P_{max}) not unbounded

$\Rightarrow$ (Fundamental Thm) (P_{min}) and (P_{max}) have optimal solutions.

(Strong Duality) Restated:

If P_{max} / P_{min} are feasible, then they have an optimal solution and their values are equal.

Possible outcomes:

P_{max}/P_{min}	Optimal	Unbounded	Infeasible
Optimal	Yes	No	No
Unbounded	No	No	Yes (WD)
Infeasible	No	Yes (WD)	Yes

2. Complementary Slackness Condition(CSC)

(P)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

(D)

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \end{aligned}$$

$$\Updownarrow$$

(P')

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax + s = b \\ & s \geq 0 \end{aligned}$$

x feasible for (P)

$\leftrightarrow s = b - Ax$
$\leftrightarrow (x, s)$ is feasible for (P')

Suppose $\bar{x}, \bar{y}$ optimal for (P), (D). $\bar{s} = (b - A\bar{x})$

$$b^T \bar{y} \underbrace{=}_{\text{primal feasibility}} (A\bar{x} + \bar{s})^T \bar{y} \underbrace{=}_{\text{dual feasibility}} (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s} = c^T \bar{x} + \bar{y}^T \bar{s}$$

If $\bar{x}, \bar{y}$ are optimal $\Rightarrow c^T \bar{x} = b^T \bar{y} \Rightarrow y^T \bar{s} = 0$

$$\bar{y}^T \bar{s} = \sum_{i \in [m]} \bar{y}_i \bar{s}_i = 0 \Rightarrow \forall i \in [m], \bar{y}_i = 0 \text{ or } \bar{s}_i = 0$$

Equivalently, if $\bar{x}, \bar{y}$ are optimal, then $\forall i \in [m], \bar{y}_i = 0$ or ith constraint is tight.

Theorem Complementary Slackness (Special Case):
If $\bar{x}, \bar{y}$ are feasible for (P), (D), $\bar{x}, \bar{y}$ are optimal $\Leftrightarrow$ CSC holds.

Complementary Slackness Condition (CSC)

(i) $\bar{y}_i = 0$

(ii) the i-th constraint of (P) is tight for $\bar{x}$

Example:

(P)

$$\begin{aligned} \max \quad & (5,3,3)x \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \end{aligned}$$

(D)

$$\begin{aligned} \min \quad & (2,4,-1)y \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} y \geq \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} \end{aligned}$$

Claim: $\bar{x} = (1,-1,1)^T, \quad \bar{y} = (0,2,1)^T$ optimal $y \geq 0$.

Pf.

$\bar{x}, \bar{y}$ feasible $\rightarrow$ check CSC!
Prove optimally by checking CSC:

(i) $\underline{\bar{y}_1 = 0}$ or $(1,2,-1)\bar{x} = 2$

(ii) $\bar{y}_2 = 0$ or $\underline{(3,1,2)\bar{x} = 4}$

(iii) $\bar{y}_3 = 0$ or $\underline{(-1,1,1)\bar{x} = -1}$

$\Rightarrow \bar{x}, \bar{y}$ optimal. $\square$

P_{max}	P_{min}
$\leq$ constraints	≥ 0
$\geq$ constraints	$\leq$ constraints
$=$ constraints	free variables
≥ 0	$\geq$ constraints
≤ 0	$\leq$ constraints
free variables	$=$ constraints

3. General CSC

$\bar{x}$, $\bar{y}$ feasible for (P_{max}), (P_{min})

(i) $\bar{x}_j = 0$, as j-th constraint of P_{min} is tight (Primal CSC)

(ii) $\bar{y}_i = 0$, as i-th constraint of P_{max} is tight (Dual CSC)

General Complementary Slackness Theorem:
$\bar{x}, \bar{y}$ feasible for P_{max}, P_{min} then these are optimal $\iff$ CSC holds.
Example:

$\boxed{P_{max}}$

$$\begin{array}{ll} \max & (-2,-1,0)x \\ \text{s.t.} & \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} x \begin{pmatrix} \geq \\ \leq \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\ & x_1 \leq 0, x_2 \geq 0 \end{array}$$

$\boxed{P_{min}}$

$$\begin{array}{ll} \min & (5,7)y \\ \text{s.t.} & \begin{pmatrix} 1 & -1 \\ 3 & 4 \\ 2 & 2 \end{pmatrix} y \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \\ & y_1 \leq 0, y_2 \geq 0 \end{array}$$

$\bar{x} = (-1,0,3)^T, \quad \bar{y} = (-1,1)^T$ are feasible and optimal.

Primal CSC:

(i) $\bar{x_1} = 0$ or $\underline{(1,-1)\bar{y} = -2}$

(ii) $\underline{\bar{x_2} = 0}$ or $(3,4)\bar{y} = -1$

(iii) $\bar{x_3} = 0$ or $\underline{(2,2)\bar{y} = 0}$

Dual CSC:

(i) $\bar{y_1} = 0$ or $\underline{(1,3,2)\bar{x} = 5}$

(ii) $\bar{y_2} = 0$ or $\underline{(-1,4,2)\bar{x} = 7}$

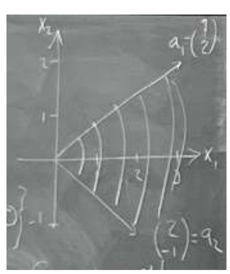
Conclusion: $\bar{x}, \bar{y}$ are optimal.

4. Visualizing Duality

Def :
$a_{11}, \ldots, a_k \in \mathbb{R}$

The cone generated by these is

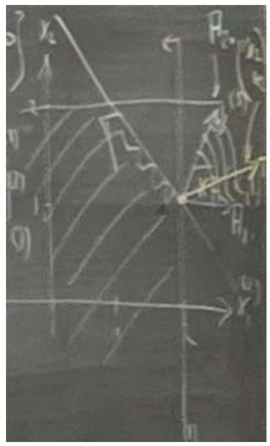
$$C = \{x : x = \sum_{i=1}^{k} \lambda_i a_i, \lambda \geq 0\}$$



$$\{x : x = \lambda_1 a_1 + \lambda_2 a_2, \lambda_1 \lambda_2 \geq 0\}$$

Example:

$$P = \left\{ x \in \mathbb{R}^2, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\}$$



Tight constraints $\bar{x}$: (1) (2)
The cone of tight constraints.

Def :

$$\delta(\bar{x}) = \{i \in [m] : A_i \bar{x} = b_i\}$$

here $\delta(\bar{x}) = \{2, 1\}$
Cone of tight constraints: $C(\bar{x}) := \{x : x = \sum_{i \in \delta(\bar{x})} \lambda_i A_i, \lambda \geq 0\}$

Theorem :
Let $\bar{x}$ be a feasible solution for $\max\{c^T x, Ax \leq b\}$, then $\bar{x}$ is optimal if and only if c is contained in the cone of tight constraints at $\bar{x}$.
Example:

P as before, then $\bar{x} = (2, 1)^T$ is optimal for $\max(\frac{3}{2}, \frac{1}{2})x$ s.t. $Ax \leq b$.

$$\exists \lambda_1, \lambda_2, \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = \frac{1}{2}$$

Pf.

$\bar{x}$ feasible and c in the cone of tight constraints at $\bar{x}$.
Need to show that $\bar{x}$ is optimal.
Strategy: Complementary Slackness.

Find dual solution $\bar{y}$ that

(1) is feasible for the dual of the given LP

(2) satisfies CSC

$\delta(\bar{x})$ index out of tight constraint at $\bar{x}$.

$$c \in C(x) \Rightarrow c = \sum_{i \in \delta(\bar{x})} \lambda_i A_i, \quad \lambda \geq 0$$

Dual of (P):

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \end{aligned}$$

CSC hold for $\bar{x}, \bar{y}$.

1. $x_j \neq 0$ j-th dual constraint tight. (Not)
2. $y_i > 0$ i-th primal constraint tight.

□

Def :

$$\bar{y}_i = \begin{cases} 0 & i \notin \delta(\bar{x}) \\ \lambda_i & i \in \delta(\bar{x}) \end{cases}$$

$$\Rightarrow \quad \underbrace{\sum_{i \in [m]} \bar{y}_i A_i}_{\lambda^T A = \sum_{i \in \delta(\bar{x})} \lambda_i A_i} \quad = c^T$$

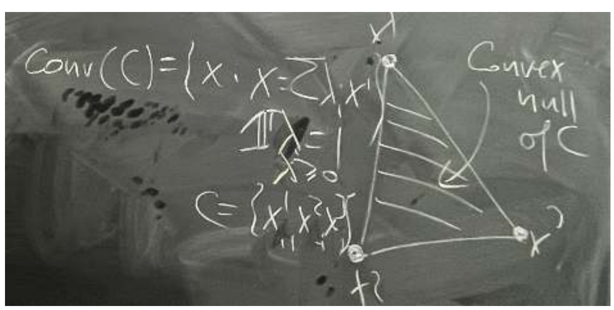
Since $\bar{y} \geq 0$, $\bar{y}$ is feasible for (D) and satisfies CSC $\Rightarrow$ $\bar{x}$ optimal and so is $\bar{x}$.

V. Integer Programming

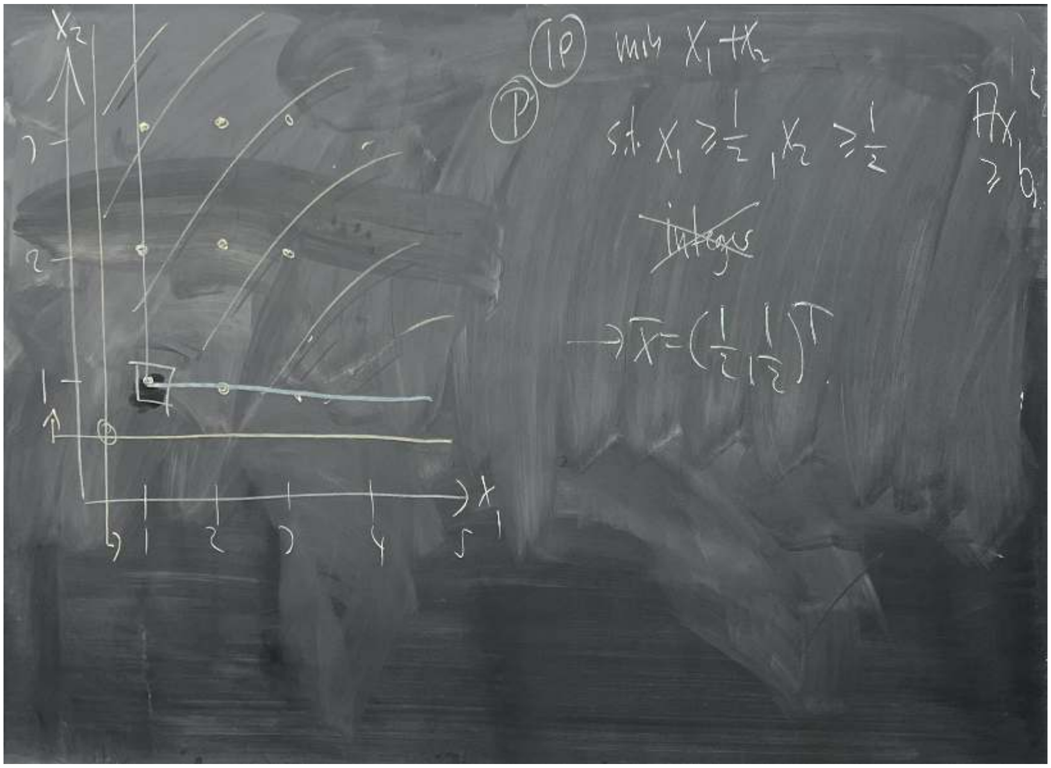
LP + integrality constraints
IPs are hard to solve sometimes.
Solution set to an IP is discrete, non-convex.
There are IPs that are feasible, bounded, but have no optimal solutions.
Def :
Convex hull $C \leq \mathbb{R}^n$

$\rightarrow$ convex hull of C is the smallest convex set C' that contains C.

$$conv(C) = \{x : x = \sum \lambda_i x_i, \quad \mathbb{1}^T \lambda = 1, \quad x \geq 0, \quad c = \{x_1, x_2, x_3\}\}$$



Convex hull is a unique, for contradiction: suppose $C_1 \neq C_2, C_1, C_2$ are both convex and $C \leq C_1, C_2$.
Example:

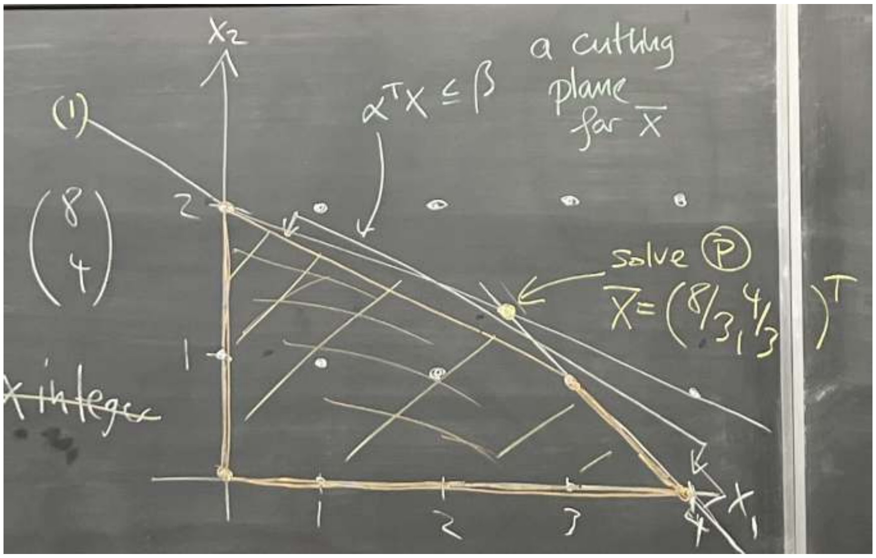


Tighten formulation: Replace $A_x \geq b$ by $A'x \geq b'$,

$$\rightarrow \quad x_1 \geq 1, x_2 \geq 1$$

As away to write the convex hull of integer solutions to IP.

i. Solving IP Problems



$$\begin{array}{ll} \max & (2,5)x \\ \text{s.t.} & \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ \text{(IP)} & x \geq 0, x \in \mathbb{Z} \\ \text{(LP)} & x \geq 0 \end{array}$$

Theorem Meyer's Theorem:
$P = \{x : Ax \leq b\}$ rational, $IP = \{x : x \in P, x \in \mathbb{Z}\}$

Then the convex hull of integer points in P can be written as a polyhedron $P' = \{x : A'x \leq b'\}$

The following hold:

1. IP defined over P is-infeasible
2. IP is unbounded iff (P') is unbounded
3. An optimal extreme solution to (P') is an optimal solution to IP

Meyer's Theorem entails one way of reducing IP to linear programming. But the reduction is not efficient.

In practice: we try to "approximate" the convex hull of integer points: odd valid constraints $\alpha^T x \leq \beta$.

s.t. all feasible integer solutions: $\bar{x}$ satisfy these constraints: $\alpha^T \bar{x} \leq \beta$

We call a valid constraint a cutting plane for fractional point $\hat{x}$ if $\alpha^T \hat{x} > \beta$

Algorithm to solve IP:

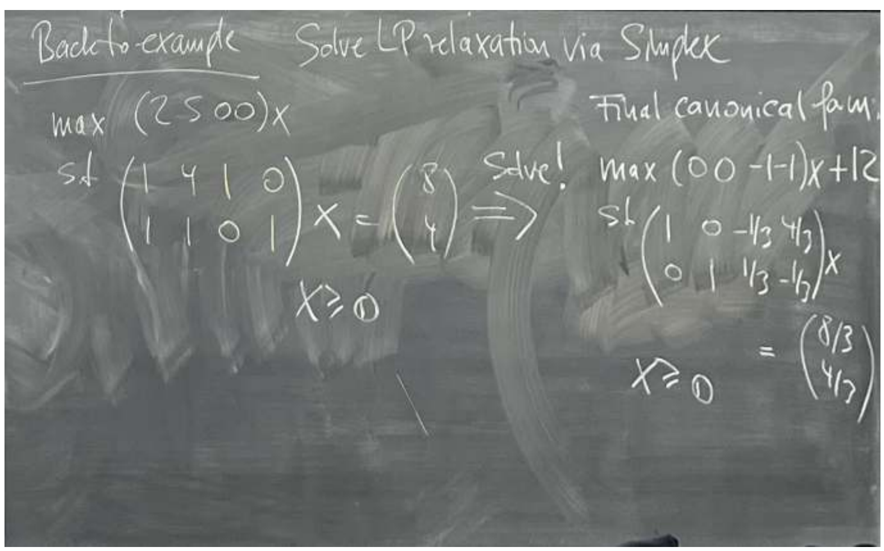
(IP)

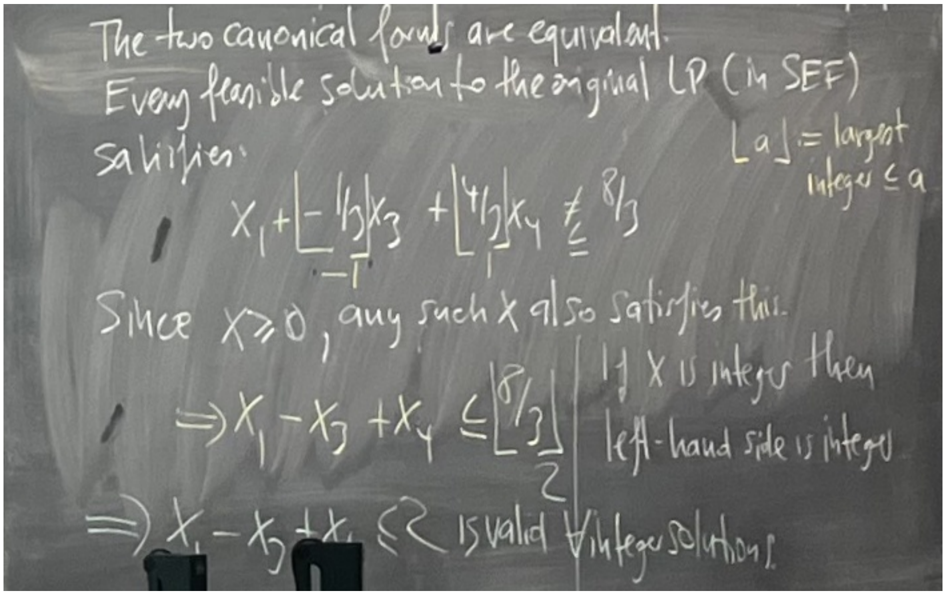
$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0, x \in \mathbb{Z} \end{aligned}$$

(P) LP relaxation of (IP)

1) Solve (P) $\rightarrow \bar{x}$
2) $\bar{x}$ is integer, then stop
3) Otherwise, find cutting plane $\alpha^T x \leq \beta$ and add to (P)

Example:





This is a cutting plane, canard basis solution is $\bar{x} = (\frac{8}{3}, \frac{4}{3}, 0, 0)^T$
$\bar{x}_1 - \bar{x}_3 + \bar{x}_4 \leq 2$

$$\begin{aligned} \max \quad & (2,5,0,0)x \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} x_1 - x_3 + x_4 &\leq 2 \quad \textcircled{\star} \\ x_3 = 8 - x_1 - 4x_2 &\geq 0 \\ x_4 = 4 - x_1 - x_2 &\geq 0 \\ \Leftrightarrow x_1 + 3x_2 &\leq 6 \\ \Rightarrow x_2 &\leq 2 - \frac{1}{3}x_1 \end{aligned}$$

Adding $\textcircled{\star}$ to LP yields convex hull of integer points.
$\Rightarrow$ Solve LP with this new constraint.

1. General cutting planes

(IP)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, x \in \mathbb{Z} \text{ (IP)} \\ & x \geq 0 \text{ (LP)} \end{aligned}$$

Solve LP (P) using simplex

$$\begin{aligned} \max \quad & \bar{c}^T x + \bar{z} \\ \text{s.t.} \quad & x_B + A_N x_N = \bar{b} \\ & x_B, x_N \geq 0 \end{aligned}$$

Let $r(i)$ be the index of the i-th basic variable.

$$\bar{x}_r(i) + \sum_{j \in N} \bar{A}_{ij} \bar{x}_j = \bar{b}_i$$

Suppose that $\bar{b_i}$ feasible (not in $\mathbb{N}$)
Valid constraint: $\bar{x}_{r(i)} + \sum_{j \in N} \lfloor \bar{A}_{ij} \rfloor x_j \leq \bar{b}_i$
(valid $\forall \bar{x} \geq 0$ that satisfy the constraints of the IP)
$\Rightarrow$ if $\bar{x}$ is integers then also, $\bar{x}_{r(i)} + \sum_{j \in N} \lfloor \bar{A}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor$ is valid.
(Note: $\bar{b}_i \notin \mathbb{N} \Rightarrow \leq \bar{b}_i$, also $\bar{x}_{r(i)} = \bar{b}_i$)

$$\begin{aligned} \max \quad & \bar{c}^T x + \bar{z} \\ \text{s.t.} \quad & x_B + \bar{A}_N x_N = \bar{b} \\ & x_{r(i)} + \sum_{j \in N} \lfloor \bar{A}_{ij} \rfloor x_j = \lfloor \bar{b}_i \rfloor \quad \text{(add another constraint)} \\ & x \geq 0 \end{aligned}$$

VI. Non-linear programming

$$\begin{aligned} \min \quad & f(x) \quad (\mathbb{R}^n \to \mathbb{R}) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad i = 1, 2, \ldots, m \\ & f(x) \leq \lambda \end{aligned}$$

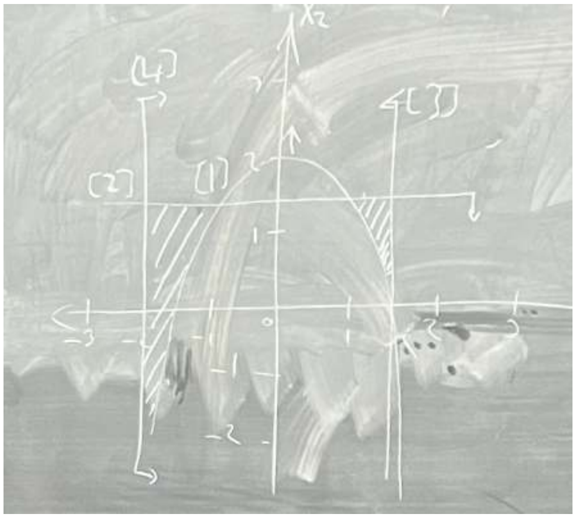
May assume that objective function if affine / linear.

We have seen:

$$x(1 - x) = 0$$

(NLP are more general than IPs)

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & -x_1^2 - x_2 + 2 \leq 0 && [1] \\ & x_2 \leq \frac{3}{2} && [2] \\ & x_1 \leq \frac{3}{2} && [3] \\ & x_1 \geq -2 && [4] \end{aligned}$$



Meta Algo

1. Find feasible $\bar{x}$
2. Is there feasible "better" point $\hat{x}$ close to $\bar{x}$?

Example:

$||\hat{x} - \bar{x}|| \leq \delta$ (fixed given constraint)
If $\bar{x} = \hat{x}$, repeat the process.
$\bar{x}$ is locally optimal $\Rightarrow$ stop.

Def :
$\bar{x}$ is locally optimal for a given NLP if $\exists \delta > 0$ s.t. $\nexists \hat{x}$ feasible with better value and $||\hat{x} - \bar{x}|| \leq \delta$.

Suppose we want to solve $\min\{c^T x : x \in S\}$, convex set S.
Proposition: If S is convex then a locally optimal $\bar{x} \in S$ is globally optimal.
Pf.

Suppose $\bar{x}$ is locally optimal but (far contradiction), assume that there is $\hat{x}$ feasible s.t. $c^T \hat{x} < c^T \bar{x}$.
Convexity: $\forall \lambda \in [0, 1]$, $\lambda \hat{x} + (1 - \lambda)\bar{x} \in S$.
Pick λ small enough s.t. $||\tilde{x} - \bar{x}|| \leq \delta$.

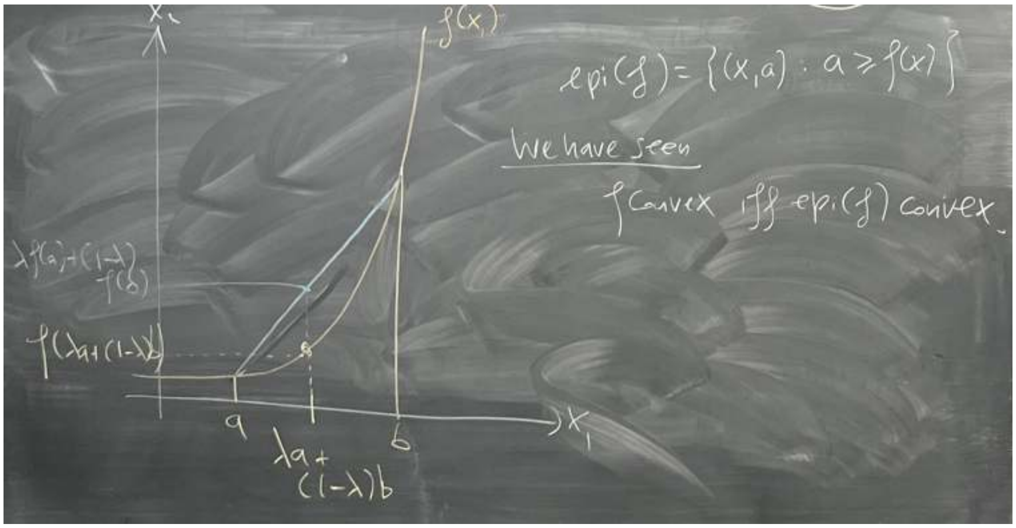
$$\begin{aligned} c^T \tilde{x} &= c^T(\lambda \hat{x} + (1 - \lambda)\bar{x}) \\ &= \lambda c^T \hat{x} + (1 - \lambda) c^T \bar{x} \\ &< \lambda c^T \bar{x} + (1 - \lambda) c^T \bar{x} \\ &= c^T \bar{x} \end{aligned}$$

Contradiction. □

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} g_i(x) \leq 0 \quad & i \in [m] \end{aligned}$$

(where $g_i : \mathbb{R}^n \to \mathbb{R}$)
Def :
$f : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f(\lambda a + (1-\lambda)b) \le \lambda f(a) + (1-\lambda) f(b) \quad \forall a, b \in \mathbb{R}^n, \lambda \in [0,1]$$



Proposition: $g : \mathbb{R}^n \to \mathbb{R}$ convex, $\Rightarrow S = \{x : g(x) \le \beta\}$ level set is convex.
Pf.

Pick $a, b \in S$ and $\lambda \in [0,1]$. $x = \lambda a + (1-\lambda)b$.
Claim: $x \in S \equiv g(x) \le \beta$.

$$g(x) = g(\lambda a + (1-\lambda)b) \le \lambda g(a) + (1-\lambda) g(b) \le \lambda \beta + (1-\lambda)\beta = \beta$$

$\to$ NLP with convex $g_i's$ have convex feasible region. □

Example:

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{s.t.} \quad & -x_2 + x_1^2 \le 0 && (1) \\ & -x_1 + x_2^2 \le 0 && (2) \\ & -x_1 + \frac{1}{2} \le 0 && (3) \end{aligned}$$

Claim: $\bar{x}$ is an optimal solution for NLP.
Strategy:

1) Relax NLP $\to$ LP.
2) Prove that $\bar{x}$ is optimal for LP, and feasible for NLP.
3) Deduce: $\bar{x}$ is optimal for NLP.

R1) Drop all constraints from NLP that are not tight at $\bar{x}$.

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{s.t.} \quad & -x_2 + x_1^2 \leq 0 \qquad (1) \\ & -x_1 + x_2^2 \leq 0 \qquad (2) \end{aligned}$$

R2) Replace non-linear constraints by linear (valid) constraints.

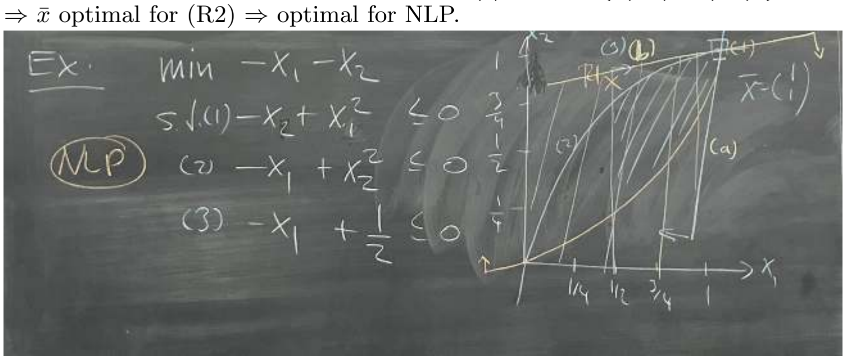
$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \leq 1 \qquad (1) \\ & -x_1 + 2x_2 \leq 1 \qquad (2) \end{aligned}$$

Claim: $\bar{x} = (1,1)^T$ is optimal for (R2).

Pf.

Both (a) and (b) are tight at $\bar{x}$. and $-c = \begin{pmatrix}1\\1\end{pmatrix} \in cone\left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}$.

$\Rightarrow \bar{x}$ optimal for (R2) $\Rightarrow$ optimal for NLP.



□

General argument: $f : \mathbb{R}^n \to \mathbb{R}$, convex, $\bar{x} \in \mathbb{R}^n$.
Then call $s \in \mathbb{R}^n$ is a subgradient of f at $\bar{x}$ if

$$f(x) \geq f(\bar{x}) + s^T(x - \bar{x}) := h(x) \quad \forall x \in \mathbb{R}^n$$

Note:

1. h is affine.
2. $h(\bar{x}) = f(\bar{x})$.
3. $h(x) \leq f(x) \quad \forall x \in \mathbb{R}^n$.

Def :
$C \in \mathbb{R}^n$ convex and $\bar{x} \in C$
$F = \{x : s^T x \leq \beta\}$ is a supporting hyperplane of C at $\bar{x}$ if

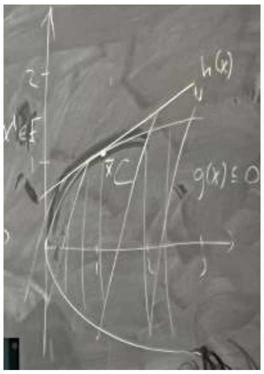
- $C \subseteq F$
- $s^T \bar{x} = \beta$

Proposition: $g : \mathbb{R}^n \to \mathbb{R}$, convex, $\bar{x} \in \mathbb{R}^n$. s.t. $g(\bar{x}) = 0$
Let s be a subgradient of g at $\bar{x}$.
Let $C = \{x : g(x) \leq 0\}$ level set of g.
Let $F = \{x : h(X) = g(\bar{x}) + s^T(x - \bar{x}) \leq 0\}$
Then F is a supporting half space of C at $\bar{x}$.
Example:

$$g(x) = x_2^2 - x_1 \bar{x} = (1,1)^T$$

Pf.

Let $x' \in C$. To show $h(x') = g(\bar{x}) + s^T(x' - \bar{x}) \leq g(x') \leq 0$.
And $h(\bar{x}) = g(\bar{x}) = 0$. □



(NLP)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & g_1(x) \leq 0 \quad \forall i \in [m] \end{aligned}$$

Suppose: $\bar{x}$ is feasible, $\forall i : g_i(\bar{x}) = 0$, also let's suppose, $g_1(\bar{x}) = 0$.
A subgradient of g_1 at $\bar{x}$, $g_1(\bar{x}) + s^T(x - \bar{x}) \leq 0$. (NLP) is a **relaxation** of NLP.
In general, let $\bar{x}$ he feasible for (NLP),
Def :
$\delta(\bar{x}) = \{i : g_i(\bar{x}) = 0\}$ set of indices of tight constraints.
Let s_i be a subgradient of g_1 at $\bar{x}$, $\forall i \in \delta(\bar{x})$.

1. Construct Relaxation of NLP

1) Drop all constraints $g_i(x) \leq 0, \forall i \notin \delta(\bar{x})$.
2) Replace $g_i(x) \leq 0$ by $g_i(x) + s_i^T(x - \bar{x}) \leq 0, \forall i \in \delta(\bar{x})$. $\Rightarrow$ (P)

$$\begin{aligned} \max \quad & -c^T x \\ \text{s.t.} \quad & s_i^T x \leq s_i^T \bar{x} - g_i(\bar{x}) \quad \forall i \in \delta(\bar{x}) \end{aligned}$$

So: (P) is a relaxation of (NLP).
Proposition: g_i are convex $\forall i$, s_i subgradient of g_i $\forall i \in \delta(\bar{x})$. $\bar{x}$ is feasible for (NLP).
If $-c \in cone\{s_i : i \in \delta(\bar{x})\}$, $\Leftrightarrow$ $\bar{x}$ is optimal for (NLP).

(NLP)

$$\begin{aligned} \max \quad & -c^T x \\ \text{s.t.} \quad & s_i^T x \leq s_i^T \bar{x} - g_i(\bar{x}) \quad \forall i \in \delta(\bar{x}) \end{aligned}$$

$\bar{x}$ feasible point, How do we know $\bar{x}$ is optimal?

Idea: find a relaxation (P) of (NLP) and show that $\bar{x}$ is optimal for (P).

1) $\delta(\bar{x}) = \{i : g_i(\bar{x}) = 0\}$.
2) Drop all inequalities that are not in $\delta(\bar{x})$.
3) Replace $g_i(x) \le 0$ by a half space (supporting half-space of g_i at $\bar{x}$).
   $\rightarrow$ use a subgradient of g_i at $\bar{x}$.

(Calculus Detour)

Fact: $f : \mathbb{R}^n \rightarrow \mathbb{R}$ a convex function, $\bar{x} \in \mathbb{R}^n$, If the gradient $\nabla f(\bar{x})$ exists then it is a subgradient.

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

evaluate at $\bar{x}$, $\nabla f(\bar{x})$

Example:

$f(x_1, x_2) = -x_2 + x_1^2$ gradient: $\nabla f(x) = (2x_1, -1)^T$

Def :
A feasible solution $\bar{x}$ for (NLP) is called a slater point if $g_i(\bar{x}) < 0$ for all $i \in \delta(\bar{x})$.

2. Kamsh, Kuhn, Tucker (KKT) Conditions

Suppose that

(i) $g_i(x)$ convex $\forall i \in [m]$.
(ii) There is a Slater point.
(iii) $\bar{x}$ is feasible, tight constraints $\delta(\bar{x})$.
(iv) $\forall i \in \delta(\bar{x})$, gradient $\nabla g_i(\bar{x})$ exists.

Then $\bar{x}$ is optimal for (NLP) iff $-c \in cone(\nabla g_i(\bar{x}), i \in \delta(\bar{x}))$.

i. Interior Point Method(IPM)

There are algorithms that, like simplex, solve LPs. Like simplex these are iterative algorithms that produce a sequence $x_1, x_2, \ldots$ of feasible points.
But: points are not on boundary of the feasible region but into interior.
Def :
$\bar{x} \in \mathbb{R}^n, \delta \in \mathbb{R}_+$

$$\underbrace{B(\bar{x}, \delta)}_{\text{Euclidean ball of radius } \delta \text{ around } \bar{x}} = \{x \in \mathbb{R}^n : \underbrace{||x - \bar{x}||_2}_{\text{Euclidean (L2) norm}} \le \delta\}$$

The interior of set $S \subseteq \mathbb{R}^n$

$$\text{int}(S) = \{x \in R^n : B(x, \delta) \subseteq S \text{ for some } \delta > 0\}$$

Call S' open if $S' = \text{int}(S)$.
Note: S' open $\iff$ S^c closed.
Consider LP in SEF:
(P)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

(D)

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y - s = c \\ & s \geq 0 \end{aligned}$$

Suppose (P) and (D) have feasible interior solutions
$\exists \bar{x} : A\bar{x} = b, \bar{x}_j > 0 \quad \forall j$
$\exists (\bar{y}, \bar{s}) : A^T \bar{y} - \bar{s} = c, \bar{s}_i > 0 \quad \forall i$
Weak Duality: $\bar{x}, \bar{y}$ feasible for (P) and (D)

$$c^T \bar{x} \leq b^T \bar{y}$$

In fact:

$$b^T \bar{y} - c^T \bar{x} = \bar{x}^T (A^T \bar{y} - c^T \bar{x}) = \bar{x}^T (A^T \bar{y} - \bar{s}) = \bar{x}^T \bar{s}$$

$\Rightarrow \bar{x}, \bar{y}$ are optimal iff $\bar{x}^T \bar{s} = 0$ (complementary slackness)

Suppose $\bar{x}^T \bar{s} > 0$, $x_1(y, s)$ optimal if

1. $Ax = b, x \geq 0$ (Primal Feasibility)
2. $A^T y - s = c, s \geq 0$ (Dual Feasibility)
3. $x_j s_j = 0 \quad \forall j$

Want: Find new feasible solution $x_1(y, s)$ s.t. $x^T s < \alpha \bar{x}^T \bar{s}, \quad \alpha \in [0, 1)$
Note: Simplex maintains (1) and (2) stuck on (3).

Replace $\bar{x}$ by $\bar{x} + dx, \quad dx \in \mathbb{R}^n$
$\bar{y}$ by $\bar{y} + dy, \quad dy \in \mathbb{R}^m$
$\bar{s}$ by $\bar{s} + ds, \quad ds \in \mathbb{R}^m$
(Note: dx, dy, ds vectors "delta" x, y, s)
s.t.

1. $A(\bar{x} + dx) = b \Leftrightarrow Adx = 0$
2. $A^T (\bar{y} + dy) - (\bar{s} + ds) = c \Leftrightarrow A^T dy - ds = 0$
3. $(\bar{x} + dx)^T (\bar{s} + ds)$ as small as possible $\Rightarrow$ quadratic.

Find ds, dx, dy s.t. (1), (2) satisfied and (3) minimized.