MATH 239 — Winter 2024: Class Notes

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January 8th 2024

Contents

I. Enumeration Combinatorics

i. Multisets versus Subsets

Typical Question: How many elements of a set S of "weight objects" have weight k?

- e.g. How many ways are there of choosing a dozen donuts from a selection of 5 different types of donuts?
- e.g. Same but there are only 3 chocolate and 2 maple left (but ≥ 12 for the others)?

Recall: A set is a collection can be distinct / distinguishable objects.

> A multiset of size 'n' of 't' types is a collection of n elements, each one of t types. Two of these elements are distinguishable if and only if they are of different types.

Then we can rewrite Q1: $\Rightarrow$ How many multisets of 4 types have size 12?

"How to describe the set of all multi sets of 4 types?"

Recall: For sets A and B

- $A \cup B$ is the set of all elements that are in A or B or both
- $A \cap B$ is the set of all elements that are in both A and B
- (Cartesian product) $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$
- (Cartesian power) $A^t = A \times A \times \cdots \times A$ (t times) $= \{(a_1, a_2, \cdots, a_t) | a_i \in A\}$
  (The sets of all ordered t-tuples of elements of A)
- We write $|A|$ for the numbers of elements in A.
- We say $A \cup B$ is a disjoint union if $A \cap B = \emptyset$

When A and B are **finite** sets, the following conditions are true:

1. $|A \cup B| + |A \cap B| = |A| + |B|$
2. $|A \times B| = |A|\,|B|$
3. $|A^t| = |A|^t$

Example:

Suppose the donut shop has in stock 3 chocolate, 5 maple, 15 lemon and 10 plain donuts.
Set of all multi sets of donuts of these 4 flavour can be described as

$$S = \overbrace{\{0,1,2,3\}}^{c} \times \overbrace{\{0,1,\dots,5\}}^{m} \times \overbrace{\{0,1,\dots,9,10\}}^{e} \times \overbrace{\{0,1,\dots,14,15\}}^{p}$$

Describe the set of all multisets of 4 flavours of donuts of size 12, that do not contain both a chocolate and a maple donut?

$S_1 = \{0,1,2,3\}\times\{0\}\times\{0,\dots,15\}\times\{0,\dots,19\}\cup\{0\}\times\{1,\dots,5\}$(Not double count for 0)$\times\{0,\dots,15\}\times\{0,\dots,19\}$

All multisets of 5 types can be described as $\mathbb{N}^4$ where $\mathbb{N} = \{0,1,2,3,\dots\}$

Claim: the number of multisets of 4 types of donuts of size 12
$\equiv$ the number of <u>subsets of size 3</u> ("3-subsets") of $\{1,\dots,15\}$ (Three points split to four group)

Every 3-subset of $\{1,\dots,15\}$ **<u>corresponds uniquely</u>** to a multi set of 4 types, size 12 $\rightarrow \{6,10,11\}$

$\underbrace{\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{5}}_{5}\ \underline{6}\ \underbrace{\underline{7}\ \underline{8}\ \underline{9}}_{3}\ \underline{10}\ \underline{11}\ \underbrace{\underline{12}\ \underline{13}\ \underline{14}\ \underline{15}}_{4} \Rightarrow (5,3,0,4)$

Conversely: start with a multi set of 4 types, size 12: (1,4,3,4)

$\underbrace{\underline{1}}_{1}\ \underline{2}\ \underbrace{\underline{3}\ \underline{4}\ \underline{5}\ \underline{6}}_{4}\ \underline{7}\ \underbrace{\underline{8}\ \underline{9}\ \underline{10}}_{3}\ \underline{11}\ \underbrace{\underline{12}\ \underline{13}\ \underline{14}\ \underline{15}}_{4} \Rightarrow \{2,7,11\}$ a 3-subset of $\{1,\dots,15\}$

This is a 1-1 correspondence, showing these two sets are the same size.

Theorem: For multi sets $0 \leq k \leq n$, the number of k-subsets of $\{1,\dots,n\}$ is $B(n,k) = \frac{n(n-1)\cdots(n-k+1)}{k!}$

<u>*Pf.*</u>

Let $\mathbb{L}$ be the set of all ordered k-tuples of distinct elements of $\{1,\dots,n\}$
Then $|\mathbb{L}| = n(n-1)\cdots(n-k+1)$

But we can describe $\mathbb{L}$ as follows: take each k-subset of $\{1,\dots,n\}$ and order it in all $k!$ different ways. The union over these is $\mathbb{L}$.

So $|\mathbb{L}| = B(n,k)k!$ # of k-subsets of $\{1,\dots,n\}$
Therefore $B(n,k) = \frac{n(n-1)\cdots(n-k+1)}{k!}$ $\square$

Example:

n = 3, k = 2, how many 2-subsets of $\{1,2,3\}$?
All 2-subsets $L: 12, 13, 21, 23, 31, 32$
$B(3,2) = \frac{3\cdot 2}{2} = 3$

ii. Binomial Coefficients

The number of k-subsets of $\{1,2,3,\dots,n\}$ is $\binom{n}{k}$ $(\frac{n(n-1)\dots(n-k+1)}{k!})$.

Which is defined for all non-negative integers k and all n (even $n \in \mathbb{C}$)

Called "Binomial coefficient"

Notes: when $n \geq k$ is an integer, $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$

- When n is an integer, $n < k$. $\binom{n}{k} = 0$ (Not enough elements to choose from)
- Note 0! $= 1$

iii. Bijection & Inverse (Recap)

Def :
A bijection from a set S to a set T is a function $f : S \to T$,

- that is injective, meaning if $f(S_1) = f(S_2)$, then $S_1 = S_2$.
- and surjective, meaning for all $t \in T$, there exists an $s \in S$ such that $f(s) = t$.

Important Property of bijection: If there exists a bijection from S to T, then $|S| = |T|$

Recall: for function $f : S \to T$ and $g : T \to S$, the **composition** function $g \circ f : S \to S$ is defined by

$$g(f(s)) = g \circ f(s) = g(f(s)), \forall s \in S$$

The function $g : T \to S$ is called the inverse of $f : S \to T$ if

- $g \circ f : S \to S$ is the identity function on S (id_S) i.e. $id_S(s) = s, \forall s \in S$
- $f \circ g : T \to T$ is the identity function on T (id_T)

Theorem :
If $f : S \to T$ has an inverse, then it is bijection.

Pf.

Suppose $f : S \to T$ has an inverse $f^{-1} : T \to S$.

Check f is **injective**: suppose $f(s_1) = f(s_2)$, for $s_1, s_2 \in S$.
Then $f^{-1}(f(s_1)) = f^{-1}(f(s_2))$,
$\Rightarrow id_S(s_1) = id_S(s_2)$,

Check f is **surjective**: take $t \in T$ arbitrary, then $f^{-1}(t) = s$ for some $s \in S$.
So $f(f^{-1}(t)) = f(s)$.
$t = id_T(t) = f(s)$, so t is in the image of f. $\square$

Example:

A permutation of a set S is a bijection from S to S. The number of permutations of a set S is $|S|!$
For a set $S = \{1, 2, 3\}$, with $|S| = 3$, there are $3! = 6$ permutations.

Example:

Let S = the set of all k-subsets of $\{1, 2, 3, \dots, n\}$, and T = the set of $0, 1$-sequences of length n, i.e. $\{0, 1\}^n$.

Claim: f: $S \to T$ is a bijection, where $f(A) = (a_1, a_2, \dots, a_n)$, where $a_i = 1$ if $i \in A$, and $a_i = 0$ if $i \notin A$.

Example:

n = 4 and k = 2
$f(\{1,2\}) = (1,1,0,0)$
$f(\{1,3\}) = (1,0,1,0)$
$f(\{1,4\}) = (1,0,0,1)$
$f(\{2,3\}) = (0,1,1,0)$
$f(\{2,4\}) = (0,1,0,1)$
$f(\{3,4\}) = (0,0,1,1)$

Verify claim:
Define $g : T \to S$ by $g(a_1, a_2, \dots, a_n) = A \subseteq \{1, 2, 3, \dots, n\}$, where $i \in A$ if and only if $a_i = 1$.
Where $a_i = 0$ or 1 for $i = 1, 2, \dots, n$.
And $g = f^{-1}$, so g is well-defined, i.e. $g(a_1, a_2, \dots, a_n)$ is in S for each $(a_1, a_2, \dots, a_n) \in T$.

From our example: $|S| = |T|$, i.e. the total number of subsets of $\{1, 2, 3, \dots, n\}$ is 2^n.

Col: For $n \geq 0$, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Pf.

Since $\binom{n}{k}$ is the number of k-subsets of $\{1, 2, 3, \dots, n\}$, and $k = 0, 1, 2, \dots, n$, are all possible sizes, we set LHS = total # subsets of $\{1, 2, 3, \dots, n\}$, which is 2^n. $\square$

Example:

Exhibit a bijection from $m(n, t)$, the set of all multisets of size with t types.
So $B(n + t - 1, t - 1)$ the set of all $(t - 1)$ subsets of $\{1, 2, 3, \dots, n + t - 1\}$.
Conclusion: $|m(n, t)| = \binom{n+t-1}{t-1}$

Define: $f : B(n + t - 1, t - 1) \to m(n, t)$ as follows: for subset $S = \{s_1, s_2, \dots, s_{t-1}\}$ with $s_1 < s_2 < \dots < s_{t-1}$,

Define $f(S) = \{m_1, m_2, \dots, m_t\}$, where $m_i = s_i - s_{i-1} - 1$ [set $s_0 = 0$ and $s_t = n + t$]

Define: $g : m(n, t) \to B(n + t - 1, t - 1)$ as follows: given $(m_1, \dots m_t)$ where $m_1 + m_2 + \dots + m_t = n$, set $s_1 = m_1 + \dots + m_i + i$ Then $1 \leq s_1 < s_2 < \dots < s_{t-1} \leq n + t - 1$
So $S = \{s_1, s_2, \dots, s_{t-1}\} \subseteq B(n + t - 1, t - 1)$ (g is inverse of f)

iv. Combinatorial Proof

Example:

Prove that for all integers $1 \leq k \leq n - 1$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

We know that the set S of all k-subsets of $\{1, \dots, n\}$
But $S = S_0 \cup S_1$, where S_0 is the set of all k-subsets of $\{1, \dots, n\}$ that do not contain n, and S_1 is the set of all k-subsets of $\{1, \dots, n\}$ that contain n.
and $S_0 \cap S_1 = \emptyset$, So $|S| = |S_0| + |S_1|$
Now S_0 is the set of all k-subsets of $\{1, \dots, n - 1\}$, so $|S_0| = \binom{n-1}{k}$
$S_1 = \{\{n\} \cup B$, where B is a k-1 subset of $\{1, \dots, n - 1\}\}$, so $|S_1| = \binom{n-1}{k-1}$
Therefore, $\binom{n}{k} = |S| = |S_0| + |S_1| = \binom{n-1}{k} + \binom{n-1}{k-1}$

Idea of combinatorial proof to show $u = v$, find a set S whose size is u_1 but viewed another way, whose size is v. Hence, $u = v$.

Example:

> Let n, k be non-negative integers. Give a combinatorial proof that $\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{i}$
> Let S be the set of all n-subsets of $\{1, \ldots, n+k\}$
> For $i = 0, 1 \ldots, k$, let S_i be the set of all n-subsets of $\{1, \ldots, n+k\}$ where the largest element is $n+i$.
> So, $S = S_0 \cup S_1 \cup \cdots \cup S_k$ and all these set are disjoint. So $|S| = |S_0| + |S_1| + \cdots + |S_k|$
> Now, $S_i = \{B \cup \{n+i\} | B$ is an n-1 subset of $\{1, \ldots, n+i-1\}\}$
> So, $|S_i| = \binom{n+i-1}{n-1}$
> Therefore $\binom{n+k}{n} = |S| = \sum_{i=0}^{k} |S_i| = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$

Theorem (Binomial Theorem):
Let n be a positive integer, let x be arbitrary.
Then $(1+x)^m = \sum_{k=0}^{m} \binom{m}{k} x^k$
Pf.

$$\begin{aligned}(1+x)^m &= \overbrace{(1+x)(1+x)\ldots(1+x)}^{m \text{ times}} \\ &= (x^0 + x^1)^m \\ &= x^{0+0+\cdots+0} + x^{0+0+\cdots+1} + \cdots + x^{1+1+\cdots+1} (2^m \text{ terms})\end{aligned}$$

The set of exponents is $\{e_1 + e_2 + \cdots + e_m : (e_1, \ldots, e_m) \in \{0,1\}^m\}$
Recall our bijection f between $\{0,1\}^m$ and the set of all subsets of $\{1, \ldots, m\}$
This f gives a bijection between

$$S_k = \text{ the set of all } k \text{ element subsets of } \{1, \ldots, m\} \text{ and } \{e_1 + e_2 + \cdots + e_m = k\}$$

Collecting together all terms with the same exponent, we get

$$(1+x)^m = \sum_{k=0}^{m} |S_k| x^k = \sum_{k=0}^{m} \binom{m}{k} x^k$$

$\square$

v. Generating Series

Let S be a set A weight function w on S is a function $w : S \to \mathbb{N} = \{0, 1, 2, \ldots\}$
Such that, for every k in $\mathbb{N}$, the set $\{s \in S : w(s) = k\}$ is finite.

Def :
The generating series for the set S with weight function w is

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$$

in the indeterminate x

Example:

1. S is a set of 12 nickels, 6 dimes, 4 quarters, and a loonie.

$$\begin{aligned} w(\sigma) &= \text{ the total value of the coins in } \sigma \\ \Phi_S(x) &= \underbrace{x^5 + \cdots + x^5}_{12} + \underbrace{x^{10} + \cdots + x^{10}}_{6} + \underbrace{x^{25} + \cdots + x^{25}}_{4} + x^{100} \\ &= 12x^5 + 6x^{10} + 4x^{25} + x^{100} \end{aligned}$$

   Note we can also write $\Phi_S(x) = \sum_{k\geq 0} a_k x^k$ where a_k is the number of elements of S of weight k (By definition of weight function, a_k is a finite non-negative integer for each k)

2. S is the set of all subsets of $\{1, \ldots, n\}$

$$w(\sigma) = |\sigma|$$

$$\Phi_S(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

3. $S = \mathbb{N}$

$$w(\sigma) = \sigma$$

$$\Phi_S(x) = 1 + x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

   (Note: This is an infinite series)

X not to be substituted by a number. It is a placeholder for the indeterminate x.

Example:

$\mathbb{N}^{\text{envn}} = \{0, 2, 4, 6, 8, \ldots\}$ with weight function $w(\sigma) = \sigma$.
Generating series $\Phi_{\mathbb{N}^{\text{even}}}(x) = 1 + 0x + +1x^2 + 0x^3 + 1x^4 + 0x^5 + \cdots = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$

vi. Formal Power Series (FPS)

Def :
A formal power series (fps) in the indeterminate x over the rational number $\mathbb{Q}$ is an expression of the form

$$A(x) = \sum_{i=0} a_i x^i$$

where $a_i \in \mathbb{Q}$ for all $i \geq 0$.

Note: Any polynomial in x with rational coefficients can be viewed as a fps where all but finitely many coefficients are 0.

vii. Operation of the fps:

- Addition for fps $A(x) = \sum_{i\geq 0} a_i x^i$ and $B(x) = \sum_{i\geq 0} b_i x^i$ is defined by

$$A(x) + B(x) = \sum_{i\geq 0} (a_i + b_i) x^i$$

- Multiplication for fps $A(x) = \sum_{i\geq 0} a_i x^i$ and $B(x) = \sum_{i\geq 0} b_i x^i$ is defined by

$$A(x) \cdot B(x) = \sum_{i\geq 0} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i$$

$$\begin{aligned} &(a_0 + a_1 x + a_2 x^2 + \ldots) \cdot (b_0 + b_1 x + b_2 x^2 + \ldots) \\ =&a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots \end{aligned}$$

  Note: $A(x)B(x) = B(x)A(x)$ by symmetry of multiplication.

  Notation: We write $[x^n]A(x)$ for "the coefficient of x^n in the fps $A(x)$"
  Example:

$[x^12](1+x+x^2+\dots) = 1$
$[x^3](1+x^2+x^4+\dots) = 0$

- Equality: $A(x) = B(x)$ means $a_i = b_i$ for every $i \geq 0$
- Substitution: Is $A(B(x))$ a valid fps?
  If the constant coefficient b_0 of $B(x)$ is zero, then $A(B(x))$ is a valid fps.
  Pf.

$$A(B(x)) = a_0 + a_1 B(x) + a_2 B(x)^2 + \dots$$

  Since $b_0 = 0$, $B(x) = b_1 x + b_2 x^2 + \dots = xC(x)$ where $C(x) = b_1 + b_2 x + \dots$

$$A(B(x)) = a_0 + a_1 x C(x) + a_2 x^2 C(x)^2 + \dots$$

  To verify $A(B(x))$ is a fps, we need $[x^i]A(B(x))$ to be a FINITE rational number for every $i \geq 0$

  For each i, $[x^i]A(B(x)) = [x^i]\underbrace{(a_0 + a_1 x C(x) + a_2 x^2 C(x)^2 + \dots + a_i x^i C(x)^i)}_{\text{This is a fps since it is a finite sum of products of fps}}$ $\square$

- Inverse: We say the fps $C(x)$ is the inverse of $A(x)$ if

$$A(x)C(x) = 1 \quad (= 1 + 0x + 0x^2 + \dots)$$

  Example:

  Claim: $A(x) = 1 + x + x^2 + \dots$ has inverse $C(x) = 1 - x$
  Check:

$$\begin{aligned}(1-x)(1+x+x^2+\dots) &= 1 - x + x - x^2 + x^2 - x^3 + \dots \\ &= 1\end{aligned}$$

  We can write $A(x) = \sum_{i \geq 0} x^i = (1-x)^{-1} = \frac{1}{1-x}$

Theorem :
Let A(x) $= \sum_{i \geq 0} a_i x^i$ be a fps. Then $A(x)$ has an inverse if and only if $a_0 \neq 0$.
Pf.

Suppose $A(x)$ has inverse $C(x) = \sum_{i \geq 0} c_i x^i$.
Then $A(x)C(x) = 1$ implies $a_0 c_0 = 1$.

Conversely, suppose $a_0 \neq 0$. Then we can write $A(x) = a_0(1 - xB(x))$ where $B(x) = -\frac{a_1}{a_0} - \frac{a_2}{a_0}x - \dots$
So $\frac{1}{1-xB(x)}$ is a valid fps, since it is $xB(x)$ substituted into $\frac{1}{1-x}$.
Hence, $A^{-1}(x)$ is the valid fps $\frac{a_0^{-1}}{1-xB(x)}$ $\square$

Example:

$$1 + x^2 + x^4 + \dots = \frac{1}{1-x^2}$$

Theorem :
Let m be positive integer. Then

$$(1-x)^{-m} = \sum_{i \geq 0} \binom{m+i-1}{m-1} x^i$$

(Negative Binomial Theorem)
Pf.

By induction on m.

Base case: m = 1

$$(1-x)^{-1} = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

Inductive step: Suppose the formula holds for m = k.
Then

$$\begin{aligned}
(1-x)^{-(m+1)} &= (1-x)^{-1}(1-x)^{-m} \\
&= (1+x+x^2+\dots)(\sum_{i\geq 0}\binom{m+i-1}{i-1}x^i) \\
&= \sum_{i\geq 0} x^i \sum_{j>0}\binom{m+j-1}{j-1}x^j \\
&= \sum_{i\geq 0}\sum_{j>0}\binom{m+j-1}{j-1}x^{i+j} \\
&= \sum_{k\geq 0}\left(\sum_{i+j=k}\binom{m+j-1}{j-1}\right)x^k \\
&= \sum_{k\geq 0}\sum_{j=0}^{k}\binom{m+j-1}{j-1}x^k
\end{aligned}$$

Lemma: We need to show that $\binom{m+k}{m} = \sum_{j=0}^{k}\binom{m+j-1}{j-1}$

$$\begin{aligned}
\binom{k+i}{i} &= \binom{k+i-1}{i} + \binom{k+i-1}{i-1} \\
&= \binom{k+i-1}{i-1} + \binom{k+i-2}{i-1} + \binom{k+i-1}{i-1} \\
&= \binom{k+i-2}{i-1} + \binom{k+i-2}{i-2} + \binom{k+i-1}{i-1} \\
&= \dots
\end{aligned}$$

The proof is completed by induction.
So we find $(1-x)^{-(m+1)} = \sum_{k\geq 0}\binom{m+k}{m}x^k = \sum_{k\geq 0}\binom{(m+1)+k-1}{(m+1)-1}x^k$ as needed.
Hence, the negative binomial theorem holds for all positive integers m. $\square$

Theorem (Finite Geometric Series):
Let k be a non-negative integer. Then $1 + x + x^2 + \cdots + x^k = \frac{1-x^{k+1}}{1-x}$

Pf.

$$\begin{aligned}
(1-x)(1+x+x^2+\cdots+x^k) &= (1+x-x+x^2-x^2+\cdots+x^k-x^k+x^{k+1}) = 1-x^{k+1} \\
&= 1-x^{k+1}
\end{aligned}$$

$\square$

Example:

What is $[x^{12}](1+x+x^2+x^3)(1+x+\cdots+x^5)(\sum_{j\geq 0} x^j)^2$?

$$\begin{aligned}
&[x^{12}]\left(\frac{1-x^4}{1-x}\cdot\frac{1-x^6}{1-x}\cdot\frac{1}{(1-x)^2}\right)\\
=&[x^{12}]\left(\frac{1-x^4-x^6+x^{10}}{(1-x)^4}\right)\\
=&[x^{12}]\left((1-x)^{-4}-x^4(1-x)^{-4}-x^6(1-x)^{-4}+x^{10}(1-x)^{-4}\right)\\
=&[x^{12}](1-x)^{-4}-[x^8](1-x)^{-4}-[x^6](1-x)^{-4}+[x^2](1-x)^{-4}\\
=&\binom{12+4-1}{4-1}-\binom{8+4-1}{4-1}-\binom{6+4-1}{4-1}+\binom{2+4-1}{4-1}\\
=&\binom{15}{3}-\binom{11}{3}-\binom{9}{3}+\binom{5}{3}\\
=&455-165-84+10=216
\end{aligned}$$

Example:

Multiplication of fps:
Find $[x^m](1+x^2)^6(-2x)^{-3}$
Using BT and NBT

$$\begin{aligned}
&(1+x^2)^6(1-2x)^{-3}\\
=&(\sum_{k=0}^{6}\binom{6}{k}x^{2k})(\sum_{l\geq 0}\binom{3+l-1}{3-1}2^l x^l)\\
=&\sum_{k=0}^{6}\binom{6}{k}x^{2k}\sum_{l\geq 0}\binom{l+2}{2}2^l x^l\\
=&\sum_{k=0}^{6}\sum_{l\geq 0}\binom{6}{k}\binom{l+2}{2}2^l x^{2k+l}\\
=&\sum_{m\geq 0}\left(\sum_{\text{pairs} 0\leq k\leq 6, l\geq 0, 2k+l=m}\binom{6}{k}\binom{l+2}{2}2^l\right)x^m\\
=&\sum_{m\geq 0}\left(\sum_{k=0}^{\min\{6,\lfloor\frac{m}{2}\rfloor\}}\binom{6}{k}\binom{m-2k+2}{2}2^{m-2k}\right)x^m
\end{aligned}$$

So answer is $[x^m](1+x^2)^6(-2x)^{-3}=\sum_{k=0}^{\min\{6,\lfloor\frac{m}{2}\rfloor\}}\binom{6}{k}\binom{m-2k+2}{2}2^{m-2k}$

viii. Generating Series for Complicated Sets

Note: Finding coefficients in fps is an essential skill in solving enumeration problems.

How to find the generating series for complicated sets?

Tools: Sum and Product Lemmas

1. Sum Lemma

Let S be a set and w a weight function on S.
Suppose $S=A_1\cup A_2$ where $A_1\cap A_2=\emptyset$.
Then the generating series for S is $\Phi_S(x)=\Phi_{A_1}(x)+\Phi_{A_2}(x)$

By definition

$$\begin{aligned}\Phi_S(x) &= \sum_{s\in S} w(s)x^s \\ &= \sum_{s\in A_1} w(s)x^s + \sum_{s\in A_2} w(s)x^s \\ &= \Phi_{A_1}(x) + \Phi_{A_2}(x)\end{aligned}$$

Example:

We saw $\Phi_{\mathbb{N}}(x) = (1-x)^{-1}$ where $w(\sigma) = \sigma$.
And $\Phi_{\mathbb{N}^{\text{even}}}(x) = (1-x^2)^{-1}$
So, $\Phi_{\mathbb{N}^{\text{odd}}}(x) = \Phi_{\mathbb{N}}(x) - \Phi_{\mathbb{N}^{\text{even}}}(x) = \frac{1}{1-x} - \frac{1}{1-x^2} = \frac{x}{1-x^2} = x + x^3 + x^5 + \dots$

Example:

Try to formulate a common generalization of Binomial Theorem and Negative Binomial Theorem, just use the definition $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$ for any $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

2. Product Lemma

Suppose C_1 and C_2 are sets with weight functions w_1 and w_2 respectively. Then the generating series for C_1C_2 with weight function w defined by

$$w(c_1c_2) = w_1(c_1) + w_2(c_2)$$

$$\Phi_{C_1C_2}(x) = \Phi_{C_1}(x) \cdot \Phi_{C_2}(x)$$

Pf.

$$\begin{aligned}\Phi_{C_1C_2}(x) &= \sum_{(C_2,C_2)\in C_1\times C_2} x^{w(C_1C_2)} \quad \text{(by definition)} \\ &= \sum_{(C_1,C_2)\in C_1\times C_2} x^{w_1(C_1)+w_2(C_2)} \\ &= \sum_{C_1\in C_1} x^{w_1(C_1)} \sum_{C_2\in C_2} x^{w_2(C_2)} \\ &= \Phi_{C_1}(x)\cdot\Phi_{C_2}(x) \quad \text{(by definition)}\end{aligned}$$

$\square$

We can easily extend the PL to any number k of sets C_1 to find $\Phi_{C_1C_2\dots C_k}(x)$ with weight function $w(c_1c_2\dots c_k) = w_1(c_1) + w_2(c_2) + \dots + w_k(c_k)$.

Example:

How many ways of choosing a dozen donuts from 4 flavors, if ≥ 12 each flavor available?
We know that $\Phi_{\mathbb{N}(x)} = \frac{1}{1-x}$ where weight function w_0 on $\mathbb{N}$ is $w_0(\sigma) = \sigma$.
Let $S = \underbrace{\mathbb{N}}_{c} \times \underbrace{\mathbb{N}}_{m} \times \underbrace{\mathbb{N}}_{l} \times \underbrace{\mathbb{N}}_{p}$
weight function on S: $w(t_1, t_2, t_3, t_4) = t_1 + t_2 + t_3 + t_4$

By PL,

$$\begin{aligned}\Phi_s(x) &= (\Phi_{\mathbb{N}}(x))^4\\ &= \left(\frac{1}{1-x}\right)^4\\ &= \sum_{i\geq 0}\binom{i+3}{3}x^i \quad \text{(by NBT)}\end{aligned}$$

Answer is $[x^{12}]\Phi_s(x) = \binom{12+3}{3} = \binom{15}{3}$

Example:

Same question, except only 3 chocolate and 5 maple are available (others ≥ 12).
Here we choose $S = \{0,1,2,3\} \times \{0,1,2,3,4,5\} \times \mathbb{N} \times \mathbb{N}$
weight function $w(t_1,t_2,t_3,t_4) = t_1+t_2+t_3+t_4$

$$\Phi_{\{0,1,2,3\}}(x) \quad \text{with respect to } w_0(\sigma) = \sigma$$

So by PL

$$\begin{aligned}\Phi_s(x) &= \Phi_{\{0,1,2,3\}}(x)\cdot\Phi_{\{0,1,2,3,4,5\}}(x)\cdot\Phi_{\mathbb{N}}(x)\cdot\Phi_{\mathbb{N}}(x)\\ &= (1+x+x^2+x^3)\cdot(1+x+x^2+x^3+x^4+x^5)\cdot\left(\frac{1}{1-x}\right)^2\\ &= \frac{1-x^4}{1-x}\cdot\frac{1-x^6}{1-x}\cdot\frac{1}{(1-x)^2}\\ &= \frac{(1-x^4)(1-x^6)}{(1-x)^4}\end{aligned}$$

Answer is $[x^{12}]\frac{(1-x^4)(1-x^6)}{(1-x)^4} = 216$

Q: Why not $S = \{0,\dots,3\}\times\{0,\dots,5\}\times\{0,\dots,12\}\times\{0,\dots,12\}$?

$$\begin{aligned}[x^{12}] &= \frac{(1-x^4)(1-x^6)(1-x^{12})^2}{(1-x)^4}\\ &= [x^{12}]\left[\frac{(1-x^4)(1-x^6)}{(1-x)^4} - \underbrace{(2x^{13}-x^{26})\frac{(1-x^4)(1-x^6)}{(1-x)^4}}_{\text{This term will go to 0 since with } x^{13} \text{ and } x^{26}}\right]\end{aligned}$$

Example:

How many ways to choose a dozen donuts if

- # chocolate is even
- # maple is odd
- there are only 2 lemon available
- there are only 4 plains available

We choose $S = \mathbb{N}^{\text{even}}_{\geq 0} \times \mathbb{N}^{\text{odd}}_{\geq 0} \times \{0,1,2\} \times \{0,1,2,3,4\}$
$w(t_1,t_2,t_3,t_4) = t_1+t_2+t_3+t_4$
With respect to $w_0(\sigma) = \sigma$ we have

$$\Phi_{\mathbb{N}^{\text{even}}_{\geq 0}}(x) = \sum_{i\geq 0} x^{2i} = \frac{1}{1-x^2}$$

$$\Phi_{\mathbb{N}_{\geq 0}^{\text{odd}}}(x) = \sum_{i\geq 0} x^{2i+1} = \frac{x}{1-x^2}$$
$$\Phi_{\{0,1,2\}}(x) = 1 + x + x^2$$
$$\Phi_{\{0,1,2,3,4\}}(x) = 1 + x + x^2 + x^3 + x^4$$

So by PL, answer is

$$\begin{aligned}
[x^{12}]\Phi_s(x) &= [x^{12}]\left(\frac{(1-x^3)(1-x^5)}{(1-x)^2} \cdot \frac{x}{(1-x^2)^2}\right) \\
&= [x^{11}]\left(\frac{1-x^3-x^5+x^8}{(1-x)^4(1+x)^2}\right) \\
&= [x^{11}](1-x)^{-4}(1+x)^{-2} - [x^8](1-x)^{-4}(1+x)^{-2} - [x^6](1-x)^{-4}(1+x)^{-2} + [x^3](1-x)^{-4}(1+x)^{-2} \\
&= 35
\end{aligned}$$

Note: How many ways of choosing 15 donuts with same restrictions?

$$[x^{15}]\Phi_s(x) = 44$$

ix. Framework for solving enumeration problems

Q: How many **z's** are of type **q**?

i) Choose a set S and a weight function w so that

$$\begin{pmatrix} \#\text{ elements of S} \\ \text{of weight q} \end{pmatrix} = \text{answer to the question}$$

ii) Find the generating series for S with respect to w, using the sum and product lemmas, etc.

iii) Find $[x^q]\Phi_S(x)$. Answer to the question.

x. Compositions

Def :

A **composition** of size n and length k is a k-tuple $(t_1, t_2, \ldots, t_k)$ of **POSITIVE** integers such that $t_1 + t_2 + \cdots + t_k = n$.

The t_i are called the part s of the composition.

How many compositions of size n like length k are there ?

Choose $S = \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1} \times \ldots \mathbb{N}_{\geq 1} = \mathbb{N}_{\geq 1}^k$

where $\mathbb{N}_{\geq 1} = \{1, 2, 3, 4, \ldots\}$

Choose w to be $w(t_1, t_2, \ldots, t_k) = t_1 + t_2 + \cdots + t_k$

Find generating series for $\mathbb{N}_{\geq 1}$ with weight function $w_0(\sigma) = \sigma$

$$\Phi_{\mathbb{N}_{\geq 1}}(x) = x + x^2 + \cdots = \frac{x}{1-x}$$

Find

$$\Phi_x(x) = \left(\Phi_{\mathbb{N}_{\geq 1}}(x)\right)^k$$
$$= \left(\frac{x}{1-x}\right)^k$$

Answer is

$$
\begin{aligned}
[x^n]\Phi_x(x) &= [x^n]\left(\frac{x}{1-x}\right)^k \\
&= [x^{n-k}]\frac{1}{(1-x)^k} \\
&= [x^{n-k}]\sum_{j=0}^{\infty}\binom{j+k-1}{k-1}x^j \\
&= \binom{k+n-k-1}{k-1} \\
&= \binom{n-1}{k-1} \text{ if } n \geq k
\end{aligned}
$$

Example:

How many compositions of size n and length k are there, where the i^{th} part is an even number $\geq 2i$, for each $1 \leq i \leq k$.

- Choose $S = \mathbb{N}^{\text{even}}_{\geq 2} \times \mathbb{N}^{\text{even}}_{\geq 4} \times \cdots \times \mathbb{N}^{\text{even}}_{\geq 2k}$
- $w(t_1, t_2, \ldots, t_k) = t_1 + t_2 + \cdots + t_k$
- For $1 \leq i \leq k$, $\mathbb{N}^{\text{even}}_{\geq 2i} = \{2i, 2i+2, 2i+4, \ldots\}$ so $\Phi_{\mathbb{N}^{\text{even}}_{\geq 2i}}(x) = x^{2i} + x^{2i+2} + x^{2i+4} + \cdots = x^{2i}(1 + x^2 + x^4 + \ldots) = \frac{x^{2i}}{1-x^2}$

So $\Phi_s(x) = \frac{x^2}{1-x^2} \cdot \frac{x^4}{1-x^2} \cdot \cdots \cdot \frac{x^{2k}}{1-x^2} = x^{2+4+\cdots+2k} \cdot \frac{1}{(1-x^2)^k}$

(Note: $2 + 4 + \cdots + 2k = 2(1 + 2 + \cdots + k) = 2k(k+1)$)

Answer is

$$
\begin{aligned}
&[x^n]\frac{x^{k(k+1)}}{(1-x^2)^k} \text{ This is 0 if } n < k(k+1) \\
=&[x^{n-k(k+1)}]\frac{1}{(1-x^2)^k} \\
=&[x^{n-k(k+1)}]\sum_{j=0}^{\infty}\binom{j+k-1}{k-1}x^{2j} \\
=&\begin{cases} 0 & \text{if } n-k(k+1) \text{ is odd or negative} \\ \binom{\frac{n-k(k+1)}{2}+k-1}{k-1} & \text{otherwise} \end{cases} \\
=&\begin{cases} 0 & \text{if n is odd or } n < k(k+1), k(k+1) \text{ is even} \\ \binom{\frac{n}{2}-\binom{k}{2}-1}{k-1} & \text{otherwise} \end{cases}
\end{aligned}
$$

Reality check: $n = 16, k = 3$

2	4	10
2	6	8
2	8	6
4	4	8
4	6	6
6	4	6

$$\binom{\frac{16}{2}-\binom{3}{2}-1}{3-1}=\binom{8-3-1}{2}=\binom{4}{2}=6$$

Example:

Show that the number of compositions of size n and length k is which each part lies in the set $\{2,\dots,8\}$ is

$$=\sum_{j=0}^{\lfloor\frac{n-2k}{2}\rfloor}\binom{k}{j}(-1)^j\binom{n-k-7j-1}{k-1}$$

Q: How many compositions of size n are there? (i.e. with all possible lengths)

Choose $S=\{\emptyset\}\cup\mathbb{N}_{\geq 1}\cup\mathbb{N}^2_{\geq 1}\cup\cdots=\mathbb{N}^*_{\geq 1}$

Choose $w(S)=w^*(a_1,a_2,\dots,a_k)=a_1+a_2+\cdots+a_k$ for each $(a_1,a_2,\dots,a_k)\in S$

Detour:

Let A be a set. We define the set A^* by

$$A^*=\cup_{i\geq 0}A^i$$

where $A^0=\{\emptyset\}$ and $A^k=A\times A\times\cdots\times A$ (k times)

Lemma: Let A be a set and w a weight function on A. Define w^* on A^* by $w^*(a_1,a_2,\dots,a_k)=w(a_1)+w(a_2)+\cdots+w(a_k)$ for all $k\geq 0$ and all $(a_1,a_2,\dots,a_k)\in A^k$. Then w^* is a weight function on A^* if and only if $w(a)\geq 1$ for every $a\in A$.

Pf.

($\Rightarrow$) If $w(a)=0$ for some $a\in A$, then $\emptyset,(a),(a,a),\dots\in A^*$ is an infinite set of elements of A^* with w^*-weight 9. So w^* is not a weight function on A^*.

($\Leftarrow$) TO show w^* is a weight function, fix a weight $n\geq 0$. If $w^*(a_1,a_2,\dots,a_k)=n$, then $w(a_1)+w(a_2)+\cdots+w(a_k)=n$. Since $w(a_i)\geq 1$ for each i, we find that $(w(a_1),w(a_2),\dots,w(a_k))$ is a composition of size n of length k.

There are finitely many of these. Also, we know $(a_1\dots a_k)\in\cup_{i=0}^n A^i$.

Since w is a weight function on A, there are finitely many elements of A with any weight $\leq n$.

So the total number of elements of A^* of weight n is finite.

Hence, w^* is a weight function on A^*. □

For a set A, we defined

$$A^*=\{\emptyset\}\cup A\cup A^2\cup A^3\cup\cdots$$

and for a weight function w on A, we defined

$$w^{*(a_1,\dots,a_n)}=w(a_1)+w(a_2)+\cdots+w(a_n)$$

for every k, $(a_1,\dots,a_n)\in A^k$.

The set of all compositions of (of all possible lengths) can be described as

$$S=\cup_{k\geq 0}\mathbb{N}^k_{\geq 1}=\mathbb{N}^*_{\geq 1}$$

where $\mathbb{N}_{\geq 1}=\{1,2,3,\dots\}$.

We consider there to be one empty composition of length 0, size 0.

To find the number of compositions of size n, we need to find the generating series $\Phi_s(x)$ for S with respect to the weight function $w^*(a_1, \ldots, a_n) = a_1 + \cdots + a_n$ and find $[x^n]\Phi_s(x)$.

By String Lemma, we know $\Phi_s(x) = \frac{1}{1-\Phi_{\mathbb{N}_{\geq 1}}(x)}$ where $\Phi_{\mathbb{N}_{\geq 1}}(x)$ is the generating series with respect to $w(\sigma) = \sigma$.

Example:

> $\mathbb{N}_{\geq 1} = \{1, 2, 3, \ldots\}$, so $\Phi_{\mathbb{N}_{\geq 1}}(x) = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$.
> So $\Phi_s(x) = \frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x} = (1-x)\sum_{n\geq 0} 2^i x^i = \sum_{n\geq 0} 2^i x^i - \sum_{n\geq 0} 2^i x^{i+1}$
> Now we find $[x^n]\frac{1-x}{1-2x} = [x^n](\sum_{n\geq 0} 2^i x^i - \sum_{n\geq 0} 2^{i-1} x^i) = \begin{cases} 1 & \text{if } n = 0 \\ 2^{n-1} & \text{if } n \geq 1 \end{cases}$

1. String Lemma

Let A be a set and w a weight function on A. if $w(a) \geq 1$ for each $a \in A$, then the generating series for A^* with respect to w^* is $\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)}$. Where $\Phi_A(x)$ is with respect to w.
Pf.

By our previous lemma we know w^* is a weight function on A^*.
Since $A^* = \cap_{k\geq 0} A^k$, we can use SL to find $\Phi_{A^*}(x) = \sum_{k\geq 0} \Phi_A(x)^k$. By PL $\Phi_{A^*}(x) = (\Phi_A(x))^k$.

$$\begin{aligned} \Phi_{A^*}(x) &= \sum_{k\geq 0} \Phi_{A^k}(x) \\ &= \sum_{k\geq 0} (\Phi_A(x))^k \\ &= \frac{1}{1-\Phi_A(x)} \end{aligned}$$

We substitute $\Phi_A(x)$ for x in $\frac{1}{1-x}$ because $\Phi_A(x)$ has constant coefficient 0. (Since no element of A has weight 0). $\square$

Example:

> Find the gs for compositions in which empty part is odd.
> We choose $S = \cup_{k\geq 0}\mathbb{N}^k_{\geq 1}$, weight function $w^*(a_1, \ldots, a_k) = a_1 + \cdots + a_k$
>
> $$S = \cup_{k\geq 0}\mathbb{N}^k_{\geq 1} = (\mathbb{N}^{odd}_{\geq 1})^k$$
>
> By String Lemma, $\Phi_s(x) = \frac{1}{1-\Phi_{\mathbb{N}^{odd}_{\geq 1}}(x)}$
> There $\Phi_{\mathbb{N}^{odd}_{\geq 1}}(x)$ is w.r.t. $w(\sigma) = \sigma$
> So $\Phi_{\mathbb{N}^{odd}_{\geq 1}}(x) = x + x^3 + x^5 + \cdots = \frac{x}{1-x^2}$
> By String Lemma, $\Phi_s(x) = \frac{1}{1-\frac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2}$ (will get solution later)

Def :
A rational function in x is an expression of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials in x. This is the standard way to express a generating series.

Example:

Find the gs for compositions in which the length is odd, and every part is odd.

$$S = \mathbb{N}_{\geq 1}^{odd} \cup (\mathbb{N}_{\geq 1}^{odd})^3 \cup (\mathbb{N}_{\geq 1}^{odd})^5 \cup \cdots$$

$$\begin{aligned}
\Phi_s(x) &= \sum_{k\geq 0} \Phi_{(\mathbb{N}_{\geq 1}^{\text{odd}})^{2k+1}}(x) \quad \text{by SL} \\
&= \sum_{k\geq 0} \Phi_{\mathbb{N}_{\geq 1}^{\text{odd}}}(x)^{2k+1} \quad \text{by PL} \\
&= \sum_{k\geq 0} \left(\frac{x}{1-x^2}\right)^{2k+1} \\
&= \left(\frac{x}{1-x^2}\right) \sum_{k\geq 0} \left(\frac{x}{1-x^2}\right)^{2k} \\
&= \frac{x}{1-x^2} \frac{1}{1-\frac{x^2}{(1-x^2)^2}}
\end{aligned}$$

(Substitution valid since $\frac{x^2}{(1-x^2)^2}$ has constant coefficient 0)

So, $\Phi_s(x) = \frac{x}{1-x^2} \frac{(1-x^2)^2}{1-2x+x^4-x^2} = \frac{x(1-x^2)}{1-3x^2+x^4} = \frac{x-x^3}{1-3x^2+x^4}$

Example:

Let a_n be the number of compositions of size n, in which all parts are odd.
So

$$\begin{aligned}
\sum_{n\geq 0} a_n x^n &= \frac{1-x^2}{1-x-x^2} \\
(1-x-x^2)\sum_{n\geq 0} a_n x^n &= 1-x^2 \\
\sum_{n\geq 0} a_n x^n - \sum_{n\geq 0} a_n x^{n+1} - \sum_{n\geq 0} a_n x^{n+2} &= 1-x^2 \\
\sum_{n\geq 0} a_n x^n - \sum_{n\geq 1} a_{n-1} x^n - \sum_{n\geq 2} a_{n-2} x^n &= 1-x^2
\end{aligned}$$

$n=0$ compare coefficients of $x^0 : LHS = a_0, RHS = 1 \Rightarrow a_0 = 1$
$n=1$ compare coefficients of $x^1 : LHS = a_1 - a_0 RHS = 0 \Rightarrow a_1 - a_0 = 0 \Rightarrow a_1 = 1$
$n=2$ compare coefficients of $x^2 : LHS = a_2 - a_1 - a_0 RHS = -1 \Rightarrow a_2 - a_1 - a_0 = -1 \Rightarrow a_2 = 1$
In general, $n \geq 3$, $x^n : a_n - a_{n-1} - a_{n-2} = 0 \Rightarrow a_n = a_{n-1} + a_{n-2}$
Recurrence relation: $a_n = a_{n-1} + a_{n-2}$

Let S_n be the set of all compositions of size n in which every part is odd.

We showed $|S_n| = a_n$ satisfies $a_0 = a_1 = a_2 = 1$ and for $n \geq 3$, $|a_n| = |a_{n-1} \cup a_{n-2}|$ using bijections.

Define: $f : S_n \to S_{n-1} \cup S_{n-2}$ for any $n \geq 3$ by $(t_1, t_2, \ldots, t_k) \in S_n$ (So, $t_1 + t_2 + \cdots + t_k = n$ all in $\mathbb{N}^{\text{odd}}$)

$$f(t_1, t_2, \ldots, t_k) = \begin{cases} (t_1, t_2, t_3, \ldots, t_k) & \text{if } t_k = 1 \\ (t_1, t_2, \ldots, t_{k-1}, t_{k-2}) & \text{if } t_k \geq 3 \end{cases}$$

define f is a bijection with every g defined as follows:

If $(t_1, t_2, \dots, t_k) \in S_{n-1}$ define $g(t_1, t_2, \dots, t_l) = (t_1, t_2, \dots, t_l, 1) \in S_n$

If $(t_1, t_2, \dots, t_k) \in S_{n-2}$ define $g(t_1, t_2, \dots, t_l) = (t_1, t_2, \dots, t_l, 3) \in S_n$

Then $f \circ g = \mathrm{id}_{S_{n-1} \cup S_{n-2}}$ and $g \circ f = \mathrm{id}_{S_n}$

xi. Binary Strings

A binary string of length n is an element of $0, 1^n$, i.e. an ordered n-tuple of bits, each of which is 0 or 1.

We will simplify notation by omitting (,,,,), so we will write $b_1 b_2 \dots b_n$ where $b_i \in 0, 1$ for each i.

e.g. 011010011110.... Length Ω

Normally we will choose the weight function for sets of binary strings to be length. We consider there to be one binary strings of length 0, written as ϵ.

Find the gs for the set S of all binary strings (With respect to length)

$$\Phi_S(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots + 2^k x^k + \cdots = \frac{1}{1 - 2x}$$

Example:

> Find the gs (w.r.t. length) for the set of binary strings with no substrings 11
> (i.e. no two consecutive 1's) e.g. 001010001001

Def :
The **concatenation** of binary strings a and b is the binary strings' ab.
For sets A and B of binary strings, the concatenation product AB is defined by $AB = \{ab | a \in A, b \in B\}$

e.g. $A = \{01, 00, 001\}$, $B = \{1, 110\}$
$AB = \{011, 01110, 001, 00110, 0011, 001110\}$ Note $|AB| = |A||B|$
e.g. $A = \{011, 01\}$, $B = \{0, 10\}$
$AB = \{0110, 01110, 010, \cancel{0}110\}$ Note $|AB| \neq |A||B|$

Def :
A concatenation product AB is said to be unambiguous ("good") if each element of AB has a unique representation as ab where $a \in A$ and $b \in B$. In this case $|AB| = |A||B|$ for all (finite) sets A,B otherwise AB is called ambiguous. ("bad")
For a set A of binary strings, we define A^* by

$$A^* = \{\epsilon\} \cup A \cup A^2 \cup A^3 \cup \cdots = \bigcup_{k=0}^{\infty} A^k (A^k\text{: concatenation product of i copies of A})$$

We say A^* is unambiguous if each element σ has a unique representation, i.e. is in exactly one A^k, and there is exactly one way choose $a_1, a_2, \dots, a_k$ in A where $\sigma = a_1 a_2 \dots a_k$

e.g. 1^* is unambiguous, $1^* = \{\epsilon, 1, 11, 111, 1111, \dots\}$

e.g. If A is a set of binary strings in which each string has a common length l, then A^* is unambiguous.

e.g. $\{001, 100, 101\}^*$ is unambiguous

$$001, 001, 100, 101, 101, 110, 101, 110 \in \{001, 100, 101\}^8$$

(Uniquely determined by the length of the strings)
Each substring of length 3 starting from the beginning of σ must define uniquely the element of A involved in the concatenation.

1. Product Lemma for binary strings

Let A and B be sets of binary strings. If A^* and B^* are unambiguous, then (w.r.t. length)

$$\Phi_{AB}(x) = \Phi_A(x)\Phi_B(x)$$

String Lemma for binary strings: For a set A of binary strings, if A^* is unambiguous, then (w.r.t. length)

$$\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$$

[Why here gs for A has 0 constant coefficients?...]

How many binary strings of length n are there, that have no substring 11?

Theorem :
The set $\{0,1\}^*$ of all binary strings can be represented $\{0,1\}^* = \{1\}^*(\{1\}^*\{0\}^*)$ and this is an unambiguous expression. (Called the "0-decomposition" of $\{0,1\}^*$)

Pf.

Let $\sigma \in \{0,1\}^n$ be an ordinary binary string then a has k 0's, for some uniquely determined $k \geq 0$.

So $\sigma = a_1 0 a_2 0 \cdots 0 a_k$ where each a_i is a (possibly empty) string of 1's.

Thus, σ is uniquely determined as the concatenation of

$$a_i \in \{1\}^* \text{ and } 0a_i \in \{1\}^*\{0\}^* \quad \text{for } 2 \leq i \leq k+1$$

$\square$

Notation: We will usually abbreviate explosions such as $\{1\}^*(\{0\}^*\{1\}^*)$ by omitting the parentheses, and writing $0_\cup 1$ instead of $0(1)$.

For $\{0,1\}$ etc. Here we would write

$$(0_\cup 1)^* = 1^*(01^*)^*$$

Such an expression is called a regular expression for formal definition

Each regular expression produce (i.e. represents) a set of binary strings. (Called a "rational language"), and the expression way or way not be unambiguous.

If an expression is unambiguous, we can apply SL, PL, String L to obtain the gs for S with respect to length.

Example:

$(0_\cup 1)^*$ be SL

- $(0_\cup 1) \to 2x$
- $(0_\cup 1)^* \to \frac{1}{1-2x}$

$1^*(01^*)^*$

- by String L $\quad 1^* \to \frac{1}{1-x}$
- PL $\quad 01^* \to \frac{x}{1-x}$
- by String L $\quad (01^*)^* \to \frac{1}{1-\frac{x}{1-x}}$

By PL $1^*(01^*)^* \to \frac{1}{1-x}\frac{1-x}{1-x-x} = \frac{1}{1-2x}$

Answer to the question from the beginning of the class:
We know all binary strings can be represented as

$$(0_\cup 1)^* = 1^*(01^*)^*$$

$$S = (\epsilon \cup 1)(0(\epsilon_\cup 1))^* \quad \text{This is unambiguous}$$

This is unambiguous because it is a restriction of the 0-decomposition of $\{0,1\}^*$
So using PL, SL, String L, the gs for S with respect to length is obtained.

- SL $\quad (\epsilon \cup 1) \to 1 + x$
- PL $\quad 0(\epsilon_\cup 1) \to x(1+x)$
- Str. L $\quad (0(\epsilon_\cup 1))^* \to \frac{1}{1-x(1+x)}$

So $\Phi_S(x) = \frac{1+x}{1-x-x^2}$ by possibly
Answer is $[x^n]\frac{1+x}{1-x-x^2}$
Question: How many binary strings of length n are there with no substring 000?
We know

$$(0_\cup 1)^* = 0^*(10^*)^* \quad \text{1-decomposition} \tag{1}$$
$$S = (\epsilon \cup 0 \cup 00)(1(\epsilon_\cup 0 \cup 00))^* \tag{2}$$

This is unambiguous since it is a restriction of the 1-decomposition.
Find the gs

- $(\epsilon \cup 0 \cup 00) \to 1 + x + x^2$
- $(1(\epsilon_\cup 0 \cup 00)) \to \frac{1}{1-x(1+x+x^2)}$ Const coefficient 0

$A : [x^n]\frac{1+x+x^2}{1-x(1+x+x^2)}$

xii. Blocks

A block in a binary string is a maximal substring consisting of all 0's or all 1's.

$$\sigma = \underbrace{0}\underbrace{1}\underbrace{00}\underbrace{111}\underbrace{000}\underbrace{1}\underbrace{0}\underbrace{111}\underbrace{000}$$

There are 5 blocks of 0's, 4 blocks of 1's.
Theorem Block Decomposition:

$$(0_\cup 1)^* = 1^*(00^*11^*)^*0^*$$

and this expression is unambiguous.
Example:

$$\sigma = (\epsilon \in 1^*)(0 \in 00^*)(1 \in 11^*)(00 \in 00^*)(111 \in 11^*)(000 \in 00^*)(1 \in 11^*)(0 \in 00^*)(111 \in 11^*)(000 \in 0^*)$$

Pf.

Let σ be a binary string, then σ start with a (possible empty) string of "1", and ends with a (possible empty) string of 0's.

So $\sigma b \subset c$ where $b \in 1^*$, $d \in 0^*$, and c alternates between a non-empty maximal string of 0's and a non-empty maximal string of 1's.

So $c = e_1 f_1 e_2 f_2 \cdots e_n f_n$ for a uniquely determined $t \geq 0$ where $e_i \in 00^*$ and $f_i \in 11^*$ for each i.

Hence, this expression for σ is unique. $\square$

Example:

Find the gs w.r.t length for the set of binary strings in which each block of 0's has even length and each block of 1's has odd length.
We know $(0 \cup 1)^* = 1^*(00^*11^*)^*0^*$, BD

$$S = (\epsilon \cup 1(11)^*0)\ (00(00)^*1(11)^*)^*\ (00)^*$$

Using SL, PL, String L:

- $\epsilon \cup 1(11)^* \to 1 + \frac{x}{1-x^2}$
- $00(00)^* \to \frac{x^2}{1-x^2}$
- $1(11)^* \to \frac{x}{1-x^2}$
- $(00)^* \to \frac{1}{1-x^2}$

Hence, we get

$$\begin{aligned}
\Phi_s(x) &= (1 + \frac{x}{1-x^2})\frac{1}{1-(\frac{x^2}{1-x^2})(\frac{x}{1-x^2})}\frac{1}{1-x^2} \\
&= \frac{1+x-x^2}{1-x^2}\frac{(1-x^2)^2}{1-2x^2+x^4-x^3}\cdot\frac{1}{1-x^2} \\
&= \frac{1+x-x^2}{1-2x^2+x^4-x^3}
\end{aligned}$$

Example:

Find the gs. w.r.t. length for the set of binary strings in which each even block of 0's is followed by exactly two 1's.
e.g. 01110100011 0011 000011 010
We know $(0 \cup 1)^* = 1^*(00^*11^*)^*0^*$,
so $S = 1^*(00(00)^*11 \cup 0(00)^*11^*)^*(\epsilon \cup 0(00)^*)$ This is unambiguous since it is a restriction of BD.
Using SL, PL, String L:

$$x^2(\frac{1}{1-x^2})x^2 + x(\frac{1}{1-x^2}) \times (\frac{x}{1-x})$$

$$\begin{aligned}
\Phi_s(x) &= \frac{1}{1-x}\frac{1}{1-(\frac{x^4}{1-x^2}+\frac{x^2}{(1-x^2)(1-x)})}(1+\frac{x}{1-x^2}) \\
&= \frac{1+x-x^2}{1-x-2x^2+x^3-x^4+x^5}
\end{aligned}$$

Example:

Find gs w.r.t. length for the set of binary strings that do not contain the substring 11100.
We know $(0 \cup 1)^* = 0^*(11^*00^*)^*1^*$, BD
so $S = 0^*(\underbrace{11^*00^* - 1111^*000^*}_{A})^*1^*$
And this is unambiguous since it is a restriction of BD.
(technically, it is not a "regular expression" but still has a clearly meaning)
By SL

$$\Phi_{11^*00^*} = \Phi_A(x) + \Phi_{1111^*000^*}(x)$$

So

$$\begin{aligned}\Phi_A(x) &= \Phi_{11^*00^*}(x) - \Phi_{1111^*000^*}(x)\\ &= \frac{x^2}{(1-x)^2} - \frac{x^5}{(1-x)^2}\end{aligned}$$

Therefore,

$$\begin{aligned}\Phi_S(x) &= \frac{1}{1-x}\frac{1}{1-(\frac{x^2-x^5}{(1-x)^2})}\frac{1}{1-x}\\ &= \frac{1}{1-2x+x^5}\end{aligned}$$

Example:

Find a recreation relation for sequence $\{a_n\}_{n\geq 0}$ of coefficients of $\Phi_S(x)$
A: $A_0 = 1$, $A_1 = 1$, $A_2 = 4$, $A_3 = 8$, $A_4 = 16$, and for all $n \geq 5$, $a_n = 2a_{n-1} - 5a_{n-5}$

xiii. Recursive Definition

1. Recursive Decomposition

A description of a set in terms of itself.

To get a generating series. Binary string $\to \{0,1\}^*$

Example:

$S \to$ Binary string,

$$S = \epsilon \cup 0S \cup 1S$$

We say that this decomposition is unambiguous, as that every string is uniquely created if we can produce every string in 1 way.
When the decomposition is unambiguous, we can use sum Lemma and Prod Lemma.

$$S(x) = x^0 + x^1S(x) + x^1S(x)$$
$$S(x)(1-x-x) = 1 \Rightarrow S(x) = \frac{1}{1-2x}$$

Example:

$B = \{\epsilon, 01, 0011, 000111, \dots\}$, $B = \epsilon \cup 0B1$
For every string in B, removing the first 0 and the last 1 yields a string in B.
Hence

$$\begin{aligned}&B(x) = x^0 + x \cdot B(x) \cdot x = 1 + x^2B(x)\\ \Rightarrow &B(x)(1-x^2) = 1\\ \Rightarrow &B(x) = \frac{1}{1-x^2}\end{aligned}$$

2. Excluded substrings

Def :
Let w and v be binary strings, we say that w contains v if $w = \Theta \cdot v \cdot \tau$ for some binary string Θ and τ.
Example:

> 01 is a substring of 11101000.

Def :
Let u, v be binary strings, we say that u avoids v. If U does not contain V as a substring.
Example:

> 111 avoids 01.

Example:

> 10101 does not avoid 010

Example:

> $S \to$ Binary strings avoiding 00.
>
> $$S = \epsilon \cup 0S \cup 1S \cup 10S$$
>
> Every string in S begins with 0 or 1. If ti begins with 0 it is the bit 0 or a string in 0S.
> Using sum lemma and product lemma.
>
> $$\begin{aligned} & S(x) = 1 + x + xS(x) + x^2 S(x) \\ \Rightarrow & S(x)(1 - x - x^2) = 1 + x \\ \Rightarrow & S(x) = \frac{1+x}{1-x-x^2} \end{aligned}$$

3. Recurrence Relations

Motivation: Let $S(x) = \sum_{n \geq 0} s_n x^n$
We want to obtain formula $S_n \to$ function of n.
Example:

> Fibonacci numbers: $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0, F_1 = 1$
>
> $$\begin{aligned} F(x) &= f_0 + f_1 x + \sum_{i \geq 2} f_i x^i && (3) \\ &= f_0 + f_1 x + \sum_{i \geq 2} (f_{i-1} + f_{i-2}) x^i && (4) \\ &= f_0 + f_1 x + x \sum_{i \geq 2} f_{i-1} x^{i-1} + x^2 \sum_{i \geq 2} f_{i-2} x^{i-2} && (5) \\ &= f_0 + f_1 x + x \sum_{i \geq 1} f_i x^i + x^2 \underbrace{\sum_{j \geq 0} f_j x^j}_{F(x)} && (6) \\ &= f_0 + f_1 x + x(F(x) - f_0) + x^2 F(x) && (7) \\ & && (8) \end{aligned}$$
>
> So,
>
> $$F(x) = 1 + x + x(F(x) - 1) + x^2 F(x)$$
> $$\Rightarrow F(x)(1 - x - x^2) = 1 + x - x \Rightarrow F(x) = \frac{1}{1-x-x^2}$$

$f_n = f_{n-1} + f_{n-2} \quad (n \geq 2)$

$f_0 = 0, f_1 = 1$

We obtained $F(x) = \frac{1}{1-x-x^2}$

We want to find f_n as a function of n.

$1 - x - x^2 = -(x^2 + x - 1)$

The roots of the polynomial are $\frac{-1 \pm \sqrt{5}}{2}$

This means that $-(x^2 + x - 1) = -(x - \alpha_1)(x - \alpha_2)$

where $\alpha_1 = \frac{-1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{-1-\sqrt{5}}{2}$

Trick: Using partial fractions

$$\frac{1}{1-x-x^2} = \frac{A}{x-\alpha_1} + \frac{B}{x-\alpha_2}$$

So,

$$1 = A(x - \alpha_2) + B(x - \alpha_1)$$

Let $x = \alpha_2 \implies 1 = A(\alpha_2 - \alpha_2) + B(\alpha_2 - \alpha_1) \implies B = \frac{1}{\alpha_2 - \alpha_1}$

Similarly, $A = \frac{1}{\alpha_1 - \alpha_2}$

Accordingly,

$$\begin{aligned}\frac{1}{1-x-x^2} &= \frac{A}{x-\alpha_1} + \frac{B}{x-\alpha_2} \\ &= \frac{A}{\alpha_1(\frac{x}{\alpha_1} - 1)} + \frac{B}{\alpha_2(\frac{x}{\alpha_2} - 1)} \\ &= \frac{-A}{\alpha_1} \sum_{n\geq 0} (\frac{x}{\alpha_1})^n + \frac{B}{\alpha_2} \sum_{n\geq 0} (\frac{x}{\alpha_2})^n\end{aligned}$$

Thus,

$$[x^n]\frac{1}{1-x-x^2} = [x^n] \sum_{n\geq 0} f_n x^n = f_n$$

But also,

$$[x^n]\frac{1}{1-x-x^2} = [x^n](*) = -\frac{A}{\alpha_1}\cdot\frac{1}{\alpha_1^n} - \frac{B}{\alpha_2}\cdot\frac{1}{\alpha_2^n}$$

Notice that

$$f_n = f_{n-1} + f_{n-2}$$

$$f_n - f_{n-1} - f_{n-2} = 0 \quad (n \geq 2)$$

The coefficients of the denominator are equal to the coefficients of the generating function.

Example:

Let g_n be the sequence defined by $g_0 = 2, g_1 = 5, g_2 = 6$, and $g_n = g_n - 3g_{n-2} - 2g_{n-3} \quad (n \geq 3)$

$$\sum_{n=3}^{\infty} g_n x^n = \sum_{n=3}^{\infty}(3g_{n-2} + 2g_{n-3})x^n$$

Hence,

$$\sum_{n=3}^{\infty} g_n x^n - \sum_{n=3}^{\infty} 3g_{n-2}x^n - \sum_{n=3}^{\infty} 2g_{n-3}x^n = 0$$

So,

$$\sum_{n=3}^{\infty} g_n x^n - 3x^2(G(x) - g_0) - 2x^3 G(x) = 0$$

So,

$$(G(x) - g_0 - g_1 x - g_2 x^2) - 3x^2(G(x) - g_0) - 2x^3 G(x) = 0$$

Factoring $G(x)$, we get

$$G(x)(1 - 3x^2 - 2x^3) = x^2(3g_0 + g_2) + g_1 x + g_0$$

$$G(x) = \frac{2 + 5x + 12x^2}{1 - 3x^2 - 2x^3}$$

Let $(g_n)_n$ be a sequence of complex numbers satisfying

$$g_n - a_1 g_{n-1} - a_2 g_{n-2} - \ldots - a_k g_{n-k} = 0 \quad (n \geq k)$$

and some $a_1, a_2, \ldots, a_k \in C$.
If $G(x) = \sum_{n\geq 0} g_n x^n = \frac{P(x)}{Q(x)}$, then the coefficients of $Q(x)$ are $a_1, a_2, \ldots, a_k$.

$$Q(x) = 1 - a_1 x - a_2 x^2 - \ldots - a_k x^k$$

Last time we get:

$$g_n - 3g_{n-2} - 2g_{n-3} = 0$$

$$\sum_{n\geq 0} g_n x^n = \frac{2 + 5x}{1 - 3x^2 - 2x^3}$$

The coefficients always match.

Motivation:

Now we start from $\sum_{n\geq 0} g_n x^n = \frac{P(x)}{Q(x)}$ and look for the initial recurrence.

Example:

> $D(x) = \sum_{n\geq 0} d_n x^n = \frac{1-3x+4x^2}{1-2x-3x^2}$
> Recurrence
> From theorem, $g_n - 2g_{n-1} - 3g_{n-2} = 0 \quad (n \geq 3)$
> We also need d_0, d_1, d_2.

Multiplying by the denominator, we get:

$$(1-2x-3x^2)D(\sum_{n\geq 0} d_n x^n) = 1 - 3x + 4x^2$$

Compare $[x^0]$ on both sides.
$1 \cdot d_0 = 1 \Rightarrow d_0 = 1$
Compare $[x']$ on both sides.
$1 \cdot d_1 - 2 \cdot d_0 = -3 \Rightarrow d_1 = -1$
Finally, compare $[x^2]$ on both sides.
$1 - d_2 - 2 \cdot d_1 = 4 \Rightarrow d_2 + 2 = 4 \Rightarrow d_2 = 2$
So, the recurrence is $d_n - 2d_{n-1} - 3d_{n-2} = 0 \quad (n \geq 3)$ with initial cases $d_0 = 1, d_1 = -1, d_2 = 2$.

xiv. Motivation

If $G = \frac{P(x)}{Q(x)}$, we can recover g_n as functions of n using
[Some notes...]
Example:

> Consider $G(x) = \frac{2+5x}{1-3x^2-2x^3}$
> Then s.t. term of the denominator is 1V
> $1 - 3x^2 - 2x^3 = (1+x) \cdot q(x)$ such that $deg(q) = 2$
> For $x = -1$, $1 - 3(-1)^2 - 2(-1)^3 = 0$
> Then we can find $g(x)$
> [some notes...]
> So,
>
> $$\begin{aligned}(-2x^3 - 3x^2 + 1) &= (1+x)(1-x-2x^2) \\ &= (x+1)(x+1)(1-2x) \\ &= (x+1)^2(1-2x) \\ &= (1-(-x))^2(1-2x)\end{aligned}$$
>
> From the theorem,
> $$g_n = P(n) \cdot (-1)^n + P_2(n) \cdot 2^n (n \geq 0)$$
> Then we can obtain $P_1(n), P_2(n), deg(P_1) < 2, deg(P_2) < 1$.
> $$g_n = (An + B)(-1)^n + C \cdot 2^n \quad (n \geq 0)$$
> For some constants A, B, C.
> For $n = 0$,
> $$\begin{aligned}g_0 &= B \cdot (-1)^0 + C \cdot 2^0 \\ &= B + C\end{aligned}$$

For $n = 1$,

$$g_1 = (A + B)(-1) + 2C$$

For $n = 2$,

$$\begin{aligned} g_2 &= (2A + B) + C \cdot 2^2 \\ &= 4C + 2A + B \end{aligned}$$

Since $G(x)$ was given as a generating function, the values are not yet known.

$$G(x) = \sum_{n \geq 0} g_n x^n = \frac{2 + 5x}{1 - 3x^2 - 2x^3} \Rightarrow (1 - 3x^2 - 2x^3)(\sum_{n \geq 0} g_n x^n) = 2 + 5x + 0x^2$$

Comparing coefficient, we get: $g_0 = 2, g_1 = 5, -3g_0 + g_2 = 0 \Rightarrow g_2 = 6$

So our system of equations is:

$$\begin{aligned} g_0 &= B + C = 2 \\ g_1 &= (A + B)(-1) + 2C = 5 \\ g_2 &= (2A + B) + 4C = 6 \end{aligned}$$

Then we can solve for A, B, C.

$$\begin{aligned} A &= -1 \\ B &= 0 \\ C &= 2 \end{aligned}$$

Review Solving a linear recursive relation for a sequence $\{a_n\}_{n \geq 0}$ means find a formula for a_n in terms of n ("closed form expression").

Example:

A sequence $\{a_n\}_{n \geq 0}$ is defined by

$$b_0 = 2, b_1 = 2, b_2 = 14, b_3 = -6, b_n - 3b_{n-1} - 3b_{n-2} + 7b_{n-3} + 6 = 0$$

Considers the polynomial $(c_y) = y^4 - 3y^3 - 3y^2 + 7y + 6$.
Find roots of c_y. $(c - 1) = 0 \rightarrow y = 1$ is a root.
Then we do long division to get $(c_y) = (y - 1)(y^3 - 4y^2 + y + 6) = (y - 1)^2(y - 2)(y - 3)$.
Solution in the recurrence is given by $b_n = (A + B_n)(-1)^n + C2^n + D3^n$.
Where $A + B_n$ polynomial in n of degree 1, $C2^n$ and $D3^n$ are exponential functions.
We find A,B,C,D by substituting $b_0 = 2, b_1 = 2, b_2 = 14, b_3 = -6$.

- $b_0 = (A + B(0))(-1)^0 + C \cdot 2^0 + D \cdot 3^0 = A + C + D = 2$.
- $b_1 = (A + B(1))(-1)^1 + C \cdot 2^1 + D \cdot 3^1 = -(A + B) + 2C + 3D = 2$.
- $b_2 = (A + 2B) + 4C + 9D = 14$
- $b_3 = -(A + 3B) + 8C + 27D = -6$.

We find

- $A = -1, B = 1, C = 1, D = 0$.

So,

$$b_n = (-1 + 4n)(-1)^n + 4 \cdot 2^n - 3^n \quad \forall n \geq 0$$

Back story:
We use the relation between the sequence $\{b_n\}_{n \geq 0}$ and the rational function $\frac{P(x)}{Q(x)}$ where the sequence is the sequence of coefficients
i.e. $\sum_{n \geq 0} b_n x^n = \frac{P(x)}{Q(x)}$.
We can find the coefficients using partial fractions.

In our example, the polynomial $Q(x) = 1 - 3x - 3x^2 + 7x^3 + 6x^4$, To use partial fractions to find the coefficients of $\frac{P(x)}{Q(x)}$, we want to write $Q(x) = (1-\lambda_1 x)^{d_1} \cdots (1-\lambda_t x)^{d_t}$ for numbers $\lambda_1 \dots \lambda_t$ and positive integers $d_1 \dots d_t$.
These inverse roots $\lambda_1 \dots \lambda_t$ are the roots of

$$\begin{aligned}(c_y) &= y^4 - 3y^3 - 3y^2 + 7y + 6\\ &= (y-\lambda_1)^{d_1} \cdots (y-\lambda_t)^{d_t}\end{aligned}$$

In our example, we have $\lambda_1 = 1, d_1 = 2, \lambda_2 = 2, d_2 = 1, \lambda_3 = 3, d_3 = 1$.
Then we will write

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-2x} + \frac{D}{1-3x} \quad \text{(using Binomial Series)}\\ &= A' \sum_{n\geq 0} (-1)^n x^n + B' \sum_{n\geq 0} (-1)^n \binom{2+n-1}{2-1} (-1)^n x^n + C \sum_{n\geq 0} 2^n x^n + D \sum_{n\geq 0} 3^n x^n\\ &= \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} \left[(-1)^n (A' + B'(n+1)) + \underbrace{2^n C + 3^n D}_{\text{poly in n degree 0}} \right] x^n\end{aligned}$$

Why is this the right polynomial for $Q(x)$?

$$\sum_{n\geq 0} b_n x^n = \frac{P(x)}{Q(x)} = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \sum_{n\geq 4} (3b_{n-1} + 3b_{n-2} - 7b_{n-3} - 6b_{n-4}) x^n$$

$$= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + 3 \sum_{n\geq 4} b_{n-1} x^n + 3 \sum_{n\geq 4} b_{n-2} x^n - 7 \sum_{n\geq 4} b_{n-3} x^n - 6 \sum_{n\geq 4} b_{n-4} x^n$$

$$\sum_{n\geq 0} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + 3x \sum_{n\geq 3} b_n x^n + 3x^2 \sum_{n\geq 2} b_n x^n - 7x^3 \sum_{n\geq 1} b_n x^n - 6x^4$$

Let $G(x) = \sum_{n\geq 0} b_n x^n$, then

$$G(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + 3x(G(x) - b_0 + b_1 x - b_2 x^2) + 3x^2(G(x) - b_0 + b_1 x) - 7x^3(G(x) - b_0) - 6x^4 G(x)$$

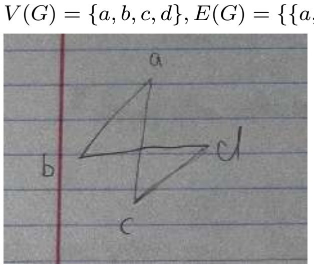
So

$$\begin{aligned}G(x)(1 - 3x - 3x^2 + 7x^3 + 6x^4) &= \text{ some poly of degree less or equal to 3, call it P(x)}\\ G(x) &= \frac{P(x)}{1 - 3x - 3x^2 + 7x^3 - 6x^4}\end{aligned}$$

II. Graph Theory

A graph G consists of a set $V = V(G)$ of vertices and a set $E = E(G)$ of edges, where each edge is a subset of V of size 2.
Example:

$V(G) = \{a, b, c, d\}, E(G) = \{\{a, b\}, \{a, c\}, \{c, d\}, \{d, b\}\}$.



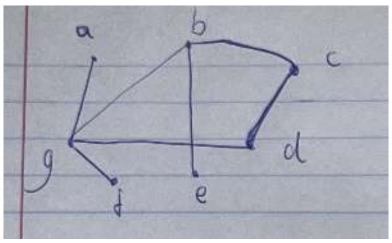
$V(G)$: set of vertex of the graph G we'll assume $V(G)$ is finite unless it is explicit stated otherwise.

$E(G)$: set pf edges pf G each edge is a subset pf $V(G)$ of size 2.

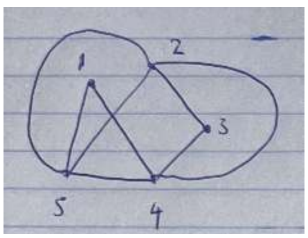
$G = (V, E)$: a graph G is a pair of sets V and E.

Example:

$V(G) = \{a, b, c, d, e, f, g\}$
$E(G) = \{\{a, g\}, \{b, g\}, \{b, e\}, \{b, c\}, \{c, d\}, \{d, g\}, \{f, g\}\}$



Example:



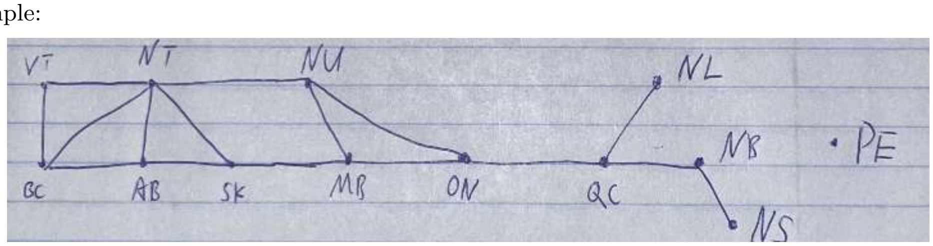
This does not represent a graph because the pair $\{2, 5\}$ appear as an edge twice.
This violates the condition that $E(G)$ is a set of pairs of vertices (It is not a multiset).
i.e. our graphs have no multiple edges. Similarly, out graphs have no loops.

Def :
We say vertices x and y are adjacent in the graph G if $xy \in E(G)$.
We also say

- the edge $e = xy$ joins x and y.
- the edge $e = xy$ has endpoints x and y.
- the edge $e = xy$ is incident to x and to y.
- x is a neighbor of y (and vice versa).

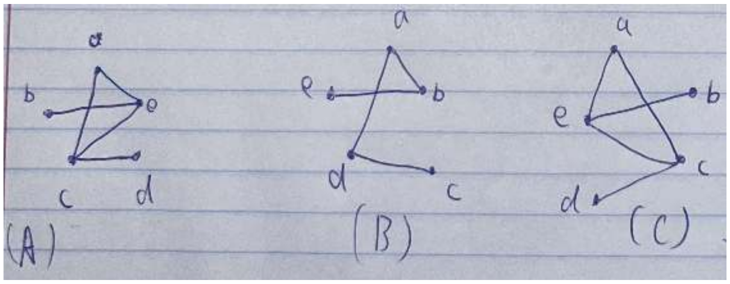
Def :
The number of edges of G incident to vertex x is called the degree of x in G_1 and is denoted $deg(x)$.

Example:



Canada graph: edges represent the relation "share a land border".
$deg(NT) = 5, deg(PE) = 0$ A vertex of degree 0 is called an isolated vertex.

Example:



$A = C$ since $V(A) = V(C)$ and $E(A) = E(C)$.
$B \neq A$ since for example $ad \in E(B)$ but $ad \notin E(A)$.
Note: B and a have the same "Structure" but just different labels on their vertices we say A and B are isomorphic.

Def :
An isomorphism from a graph A to a graph B, is a function $f : V(A) \to V(B)$ that is

1. a bijection from, $V(A)$ to $V(B)$
2. satisfies $f(x)f(y) \in E(B)$ if and only if $xy \in E(A)$.

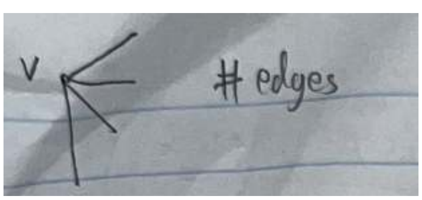
Example:

Follow the last graph:
$f : V(A) \to V(B)$ in our example is an isomorphism:

- $f(a) = a$
- $f(b) = e$
- $f(c) = d$
- $f(d) = c$
- $f(e) = b$

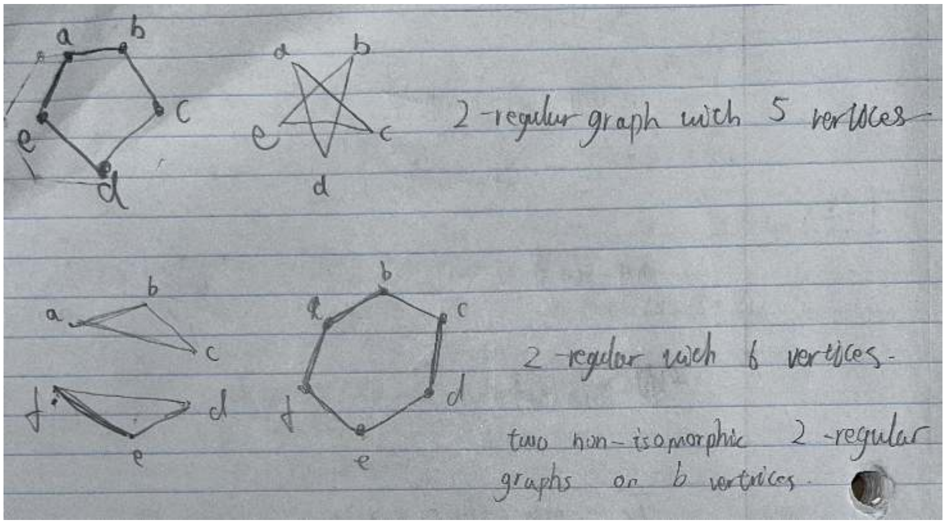
Check edges $ac \in E(A)$ and $f(a)f(c) = ed \in E(B)$.

Def :
The degree $deg(x)$ of a vertex x in a graph G is the number of edges incident to v.



Def :
If every vertex in G has the same degree r, we say G is r-regular.

Example:



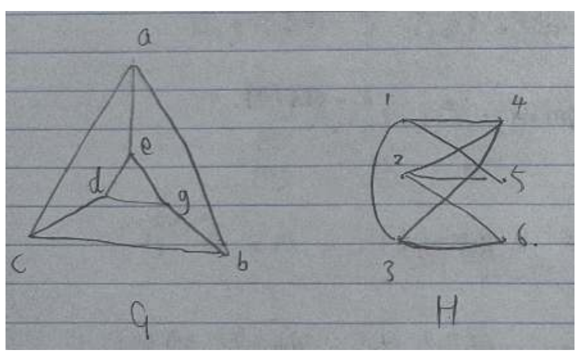
Q: Draw all 2-regular graphs with 5 vertices up to isomorphism [Means: every 2-regular graph with 5 vertices must be isomorphic to exactly one graph on your list.]
Not possible!

Theorem :
For any graph G, $\sum_{v\in V(G)} deg(v) = 2|E(G)|$.

Pf.

The LHS counts every edge of G exactly twice, once for each of its endpoints. $\square$

Corollary:(Handshaking Lemma): In any graph G, the number of vertices of odd degree is even.



Claim: G and H are isomorphic we must exhibit an isomorphism $f : V(G) \to V(H)$.

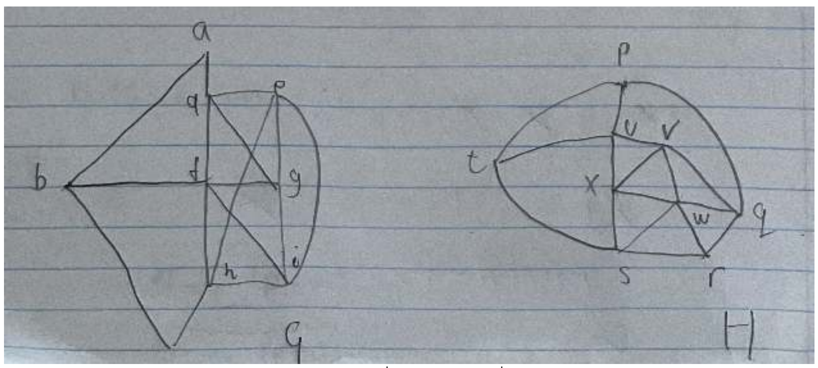
v	f(v)
a	4
b	3
c	1
d	5
e	2
f	6

Check edges of G:

- $ab \to 43$
- $bc \to 31$

- $ca \to 14$
- $eg \to 26$
- $gd \to 65$
- $de \to 52$
- $ae \to 42$
- $gb \to 63$
- $dc \to 51$

There are no other edges of H, These vertices f is an isomorphism.
Q: How to show two graphs are isomorphic?
A: Exhibit and isomorphism.
Q: How to show two graphs are not isomorphic?
A: We don't know lol, no algorithm for polynomial time. Try: looking for "substructures" that are independent of the vertex labels, and are different in the two graphs.
Example:



degree 3	4	5
a b c	d e g i h	f

Degree sequence: 3,3,3,4,4,4,4,4,5

degree 3	4	5
p t r	u v q s x	w

Degree sequence: Same sequence as G.

IF there exists an isomorphism from G to H then it must map$\{a, b, c\}$ of degree 3 in G to $\{p, t, r\}$ of degree 3 in H.

e.g

- $a \to p$
- $b \to t$
- $c \to r$

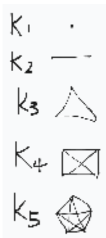
Would not work, since $bc \in E(G)$ but $tr \notin E(H)$.

More generally, no map from $\{a, b, c\}$ to $\{p, t, r\}$ can work.

Can be part of an isomorphism, since a, b, c are all mutually adjacent in G. (i.e. they form a triangle) but r is not adjacent to either p or t in H.

This certifies that G and H are not isomorphic.

i. Some special graph

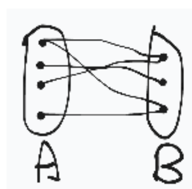


Complete graph K_p on p vertices all $v, w \in V(K_p)$ the edge $vw \in E(K_p)$.

So $|E(K_p)| = \binom{p}{2} = \frac{p(p-1)}{2}$.

Normally we don't label the vertices of K_p.

(All complete graphs with p vertices are isomorphic.)



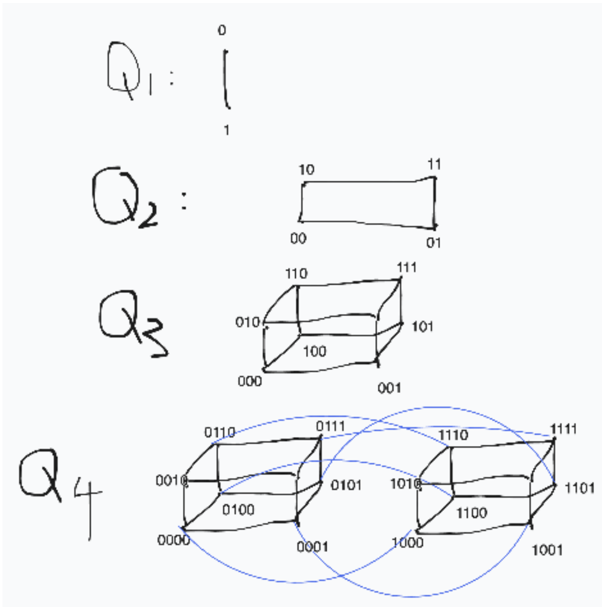
- We say a graph G is bipartite if there exists a partition $A \cup B$ of $V(G)$ such that every edge of G has exactly one vertex in A and exactly one vertex in B.
- (A special bipartite graph) The complete bipartite graph $K_{p,r}$ with parameters p and r is the bipatite graph with vertex classes A and B, $|A| = p$, $|B| = r$, and all edges joining a vertex in A to a vertex in B.

Lemma: Let G be a bipartite graph with vertex classes A and B.

Then
$$\sum_{a \in A} deg(a) = \sum_{b \in B} deg(b) = |E(G)|$$

Pf.

Both expressions count every edge of G exactly once. $\square$



For each n, we define the n-cube Q_n to be the graph with vertex set $V(Q_n) = \{\sigma \in \{0,1\}^n\}$ (binary strings of length n)

The edge set if $\{\sigma\tau : \sigma, \tau \text{ differ in exactly one bit}\}$

- Q_n has 2^n vertices
- every vertex of Q_n has degree n, i.e. Q_n is n-regular

$$\text{So by } \begin{cases} \text{Handshaking Lemma:} & 2|E(Q_n)| = n|V(Q_n)| \\ \text{Double Counting:} & 2|E(Q_n)| = n2^{n-1} \end{cases}$$

Lemma: Q_n is bipartite for every n.

Pf.

We can partition $V(Q_n)$ into

$$A = \{\sigma \in \{0,1\}^n : \sigma \text{ has an odd number of 1's}\}$$

$$B = \{\sigma \in \{0,1\}^n : \sigma \text{ has an even number of 1's}\}$$

Then each edge $\sigma\tau$, σ and τ differ in exactly one bit, so one of σ, τ has an even number of 1's, the other has an odd number of 1's. □

Let n and k be fixed, $1 \leq k \leq n$. The kneser graph with parameters n and k has vertex set $\{U \leq \{1, \dots, n\}, |A| = k\}$ The edge set is

$$\{u \leq \{1, \dots, n\}, |A| = k\}$$

$$\{uw : u \cap w = \emptyset\}$$

This graph has $\binom{n}{m}$ vertices. (If $k > \frac{n}{2}$, then this graph has no edges.)

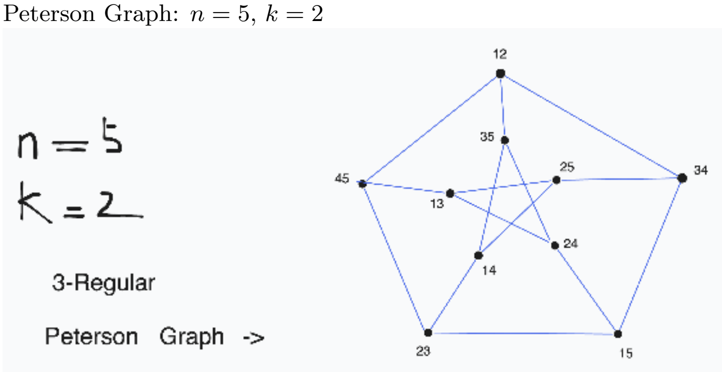
The edges of a vertex U is the kaneser graph with parameters n and k is $[1, 2, 3, \ldots, n]$

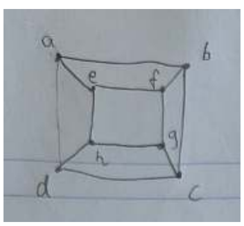
$$\binom{n-k}{k}$$

Example:

> Draw all bipartite graphs with 4 vertices up to isomorphism. (i.e. every bipartite graph with 4 vertices should be isomorphic to exactly one graph in our list.)

Peterson Graph: $n = 5$, $k = 2$





Def :
A walk in a graph G from a vertex x to a vertex y is a sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$ where $v_0 = x, v_k = y$ and for each $i \geq 1$, $e_i = v_{i-1}v_i$ often we just write the sequence of vertices $v_0 v_1 \cdots v_k$, since the edges are determined by the vertices.

Example:

> a walk from d to g:
> $dheabfehg$

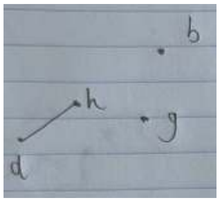
Def :
A path in a graph G is a walk in which all vertices are distinct, and hence all edges are distinct.

The length of a walk or a path is the number of edges (counted with multiplies for walks). A single vertex is a path (or walk) of length 0.
Def :
A subgraph of a graph G is a graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq \{x, y \in E(G) : x, y \in V(H)\}$
If $E(H) = \{xy \in E(G) : x, y \in V(H)\}$ then we way it is the subgraph of G induced by $V(H)$

If $P = v_0 v_1 \cdots v_k$ is a path in a graph G then P determines a subgraph of with vertex set $\{v_0, v_1, \ldots, v_{k-1}, v_k\}$. We refer to the subgraph as a path in G as well.

Def :
The subgraph h of G is said to be spanning if $V(H) = V(G)$

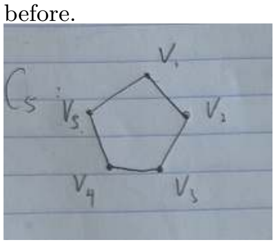
Example:



H a subgraph of G with vertex set $\{d, h, g, b\}$ (not induced)

Q: If H is a spanning induced subgraph of G, then what is H?
$H = G$
Does our examples G have a spanning path? A spanning path is called a Hamilton path
Def :
A cycle in a graph G is a subgraph with vertex set $\{v_1, v_2, \ldots, v_k\}$ where $k \geq 3$, and $v_{i-1}v_i$ is an edge for each $i \geq 2$ and $v_1 v_k$ is an edge.
The length of the cycle is k

We also talk about a cycle c_k as a graph by itself, with vertex set $\{v_1 \ldots v_k\}$ and edge set as described before.

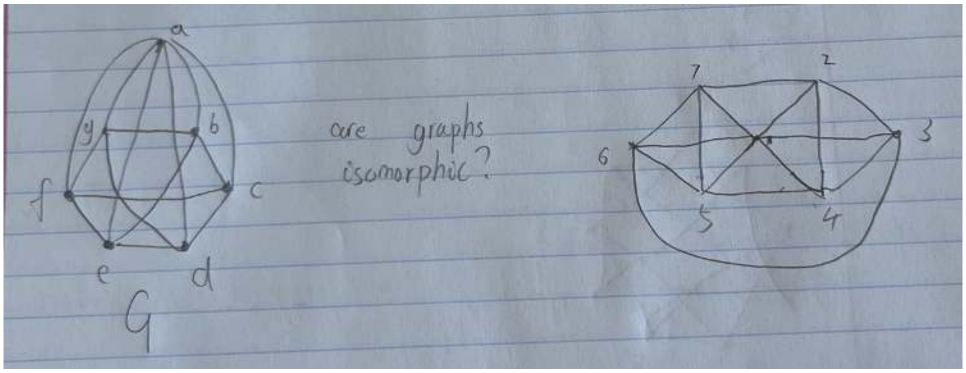


A cycle of length 4 in G: $abcd$
A cycle of length 6: $aefghd$
A cycle of length 8: $aefbcghd$ a spanning cycle (Called a Hamilton cycle)

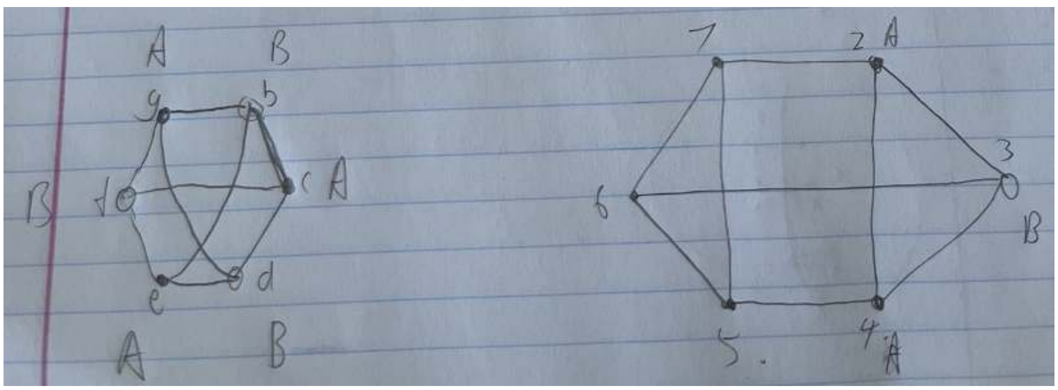
Q: How to tell whether two graphs are isomorphic?
If yes: must exhibit an isomorphism
If NO: no good procedure known for this!
Try: comparing basic properties of the two graphs, such as

- degree sequence
- cycle lengths
- bipartiteness
- presence of special sub-graphs

Note: If the two graphs differ in any of these respects, they are NOT isomorphic.
Example:



Are they isomorphic?
Degree sequence:
G: $6, 4, 4, 4, 4, 4, 4$ H: $5, 5, 5, 5, 5, 5, 5$
If there exists an isomorphism $h : V(G) \to V(H)$ then $h(a) = 1$
This would mean that G', the graph obtained by removing a from G, would be isomorphic to H', the graph obtained by removing 1 from H.

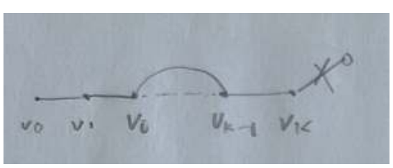


H' contains a cycle of length 3 : 2, 3, 4, G' has no cycle of length 3, OR G' is bipartite and H' is not.

Lemma: Let G be a (nonempty) graph in which every vertex has degree at least 2. Then G contains a cycle.

Pf.

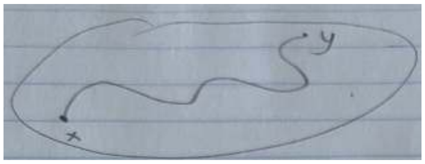
Let P be the longest path in G, say $P = v_0 v_1 \cdots v_k$. Then every neighbor of v_k is in $\{v_0, \cdot, v_{k-1}\}$, since otherwise P could be extended to a longer path.



Since $deg(v_k) \geq 2$, there exists $v_i, 0 \leq i \leq k-2$, such that $v_i v_k \in E(G)$. Then $v_i v_{i+1} \cdots v_k$ gives a circle in G. $\square$

Def :

A graph G is said to be <u>connected</u> if for every $x, y \in V(G)$, there exists a path from x to y. If G is not connected it is called <u>disconnected</u>.



<u>Def</u> :
A <u>component</u> of a graph G is a max connected subgraph of G.

Means: H is a component of G $\Leftrightarrow$

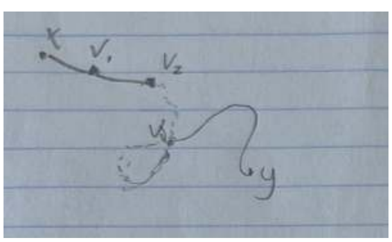
- H is a subgraph of G
- H is connected
- there is no larger connected subgraph J of G containing H

Lemma: Suppose there exists a walk from x to y in a graph G. then there exists a path from x to y in a graph G.

<u>*Pf.*</u>

Let $w = v_0 v_1 \cdots v_k$ be a shortest walk from $x = v_0$ to $y = v_k$ in G. We claim W is a path from x to y. If not, then $v_i = v_j$ for some $i < j, 0 \leq i, j \leq k$.

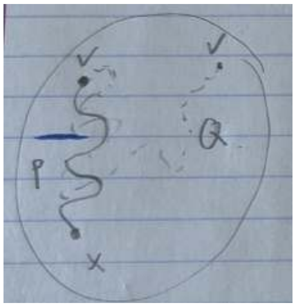
But then $v_0 v_1 \cdots v_i v_{j+1} \cdots v_k$ would be a shorter walk from x to y, contradicting our assumption on w. Here w is a path.



□

<u>**Theorem :**</u>
A (nonempty) graph G is connected if and only if there exists a vertex v in G such that for all $x \in V(G)$ there is a path from x to v.

<u>*Pf.*</u>



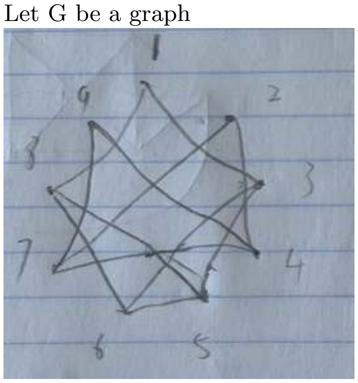
(⇒) Suppose G is connected. Fix any vertex v. Then by definition of connected, every vertex x is joined by a path to v.

(⇐) Suppose v exists. Let $x, y \in V(G)$. Then there exists a path P from x to v and a path Q from y to v. So $P\bar{Q}$, the walk obtained by following P from x to v, followed by $\bar{Q}$, the path from y to v taken in reverse, is a walk from x to y, Hence by the lemma, there exists a path from x to y. So G is connected. □

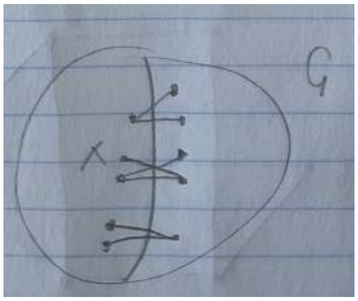
Example:

> Show the n-cube Q_n is connected $V(Q_n) = \{0,1\}^n$. Choose $v = 00\cdots0$. Take an arbitrary vertex x. Let $i_1 \cdots i_k$ be the positions of the 1's in x. For $j = 1 \ldots k$, Let x_I be the binary string with q in positions $i_1 \cdots i_j$ and 0's elsewhere.
> Then
>
> $$\begin{gathered} vx_1 \in E(Q_n) \\ x_1x_2 \in E(Q_n) \\ \vdots \\ x_{k-1}x_k \in E(Q_n) \quad x_k = x \end{gathered}$$
>
> Therefore, $vx_1x_2\cdots x_{k-1}x_k$ is a path from v to x. Hence, by the theorem, Q_n is connected.

How to show a graph is not connected?
Let G be a graph

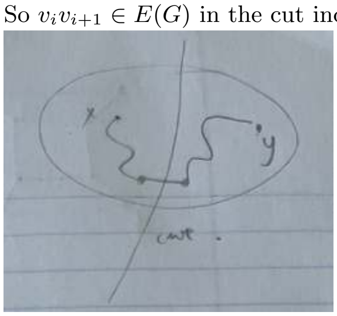


Def :
Let X be a subset of $V(G)$. The cut induced by X is $\{xy \in E(G) m x \in X, y \notin X\}$ We say $X \subseteq V(G)$ is proper if $\emptyset \neq X \neq V(G)$.

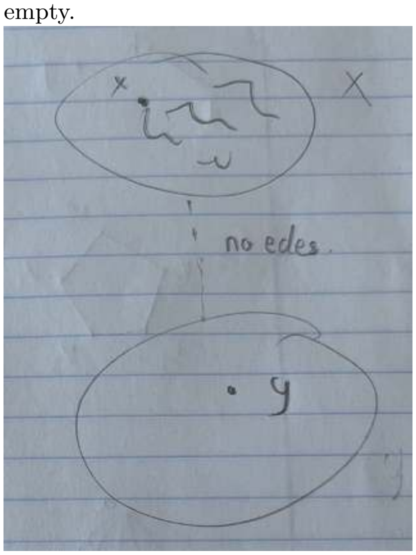


Theorem :
Let G be a graph then G is connected if and only if for every proper subset X of $V(G)$ the cut induced by X is non-empty.
Pf.

($\Rightarrow$) Assume G is connected. Let X be a proper subset of $V(G)$. Since X is proper, there exist vertices $x \in X$ and $y \notin X$.
Let $P = v_0 v_1 \cdots v_k$ be a path from x to y, where $v_0 = x$ and $v_k = y$. Let i be the smallest index such that $v_i \in X$ and $v_{i+1} \notin X$. (Exists since $v_0 \in X, v_k \notin X$)
So $v_i v_{i+1} \in E(G)$ in the cut induced by X.



($\Leftarrow$) Suppose G is not connected. Then there exists $x, y \in V(G)$ that are not joined by a path. Let X be the component $0 \in G$ that contains x. Then $y \notin X$. By definition of component, the cut induced by X is empty.



□

Example:

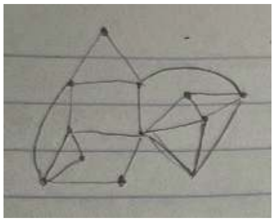
Let a_n be the graph with vertex set $\{0,1\}^n$, where $\sigma\sigma' \in E(a_n)$ if and only if σ and σ' differ in exactly 2 bits.

Example:

Prove G_n is not connected, by showing

$$X = \{\sigma \in V(G_n) : \sigma \text{ has an even number of 1's}\}$$

induces an empty cut.



Def :
A walk $v_0v_1 \dots v_n$ is a closed walk if $v_0 = v_n$.

Note that if $v_0v_1 \dots v_n$ is a closed walk, then for each i, $v_iv_{i+1} \dots v_i$ is also a closed walk.

Def :
An Eulerian walk in a graph G is a closed walk that contains every edge of G exactly once.

Def :
A graph G is said to be even if every vertex has even degree.

Theorem :
Let G be a connected graph. Then G has an Eulerian circuit if and only if G is even.

Pf.

($\Rightarrow$)

If G has an Eulerian circuit W then the edges at each vertex must be paired by w as it enters and leaves v on distinct edges. Thus, $deg(v)$ is even.

($\Leftarrow$)

Suppose G is a connected even graph. We preceded by induction on $m = |E(G)|$.

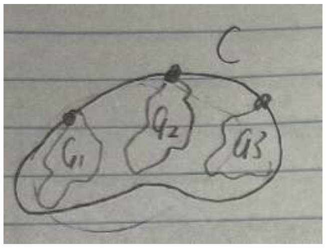
Base case: $m = 0$. Then G is a single vertex and the result is trivial.

Hypothesis: Assume that the result holds for all connected even graphs with fewer than m edges has an Eulerian circuit.

Induction Step: Consider a connected even graph G with m edges. Hence, every vertex of G has degree ≥ 2. Hence, by a previous theorem, G contains a cycle C.

Let G' be the graph obtained by removing all edges of C from G. Then in G',

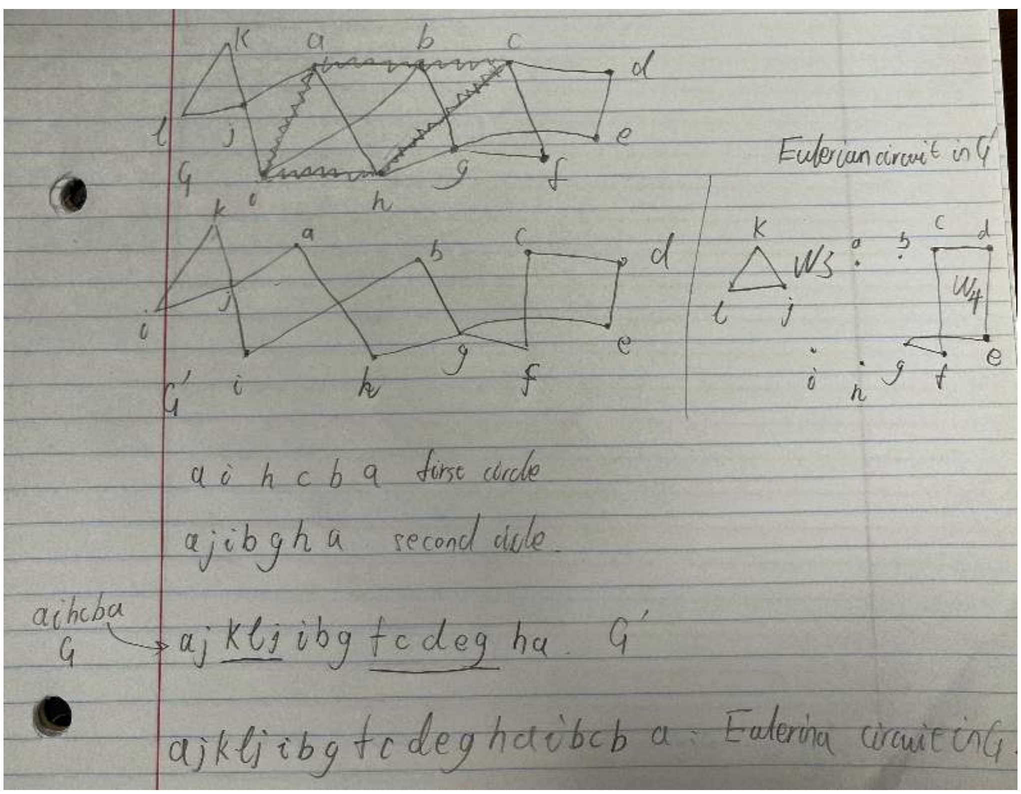
$$deg_{g'}(v) = \begin{cases} deg_G(v) - 2 & \text{if } v \in V(C) \\ deg_G(v) & \text{otherwise} \end{cases}$$



Then G' is even, so each component G_i of G' is connected, even, and has fewer than m edges. Hence, by IH G_i has an Eulerian circuit W_i. Each G_i contains a vertex v_i of C, since G was connected.

Thus inserting into C, W_i at v_i for each i, gives an Eulerian circuit of G. $\square$

Example:



ii. Bridges and Trees

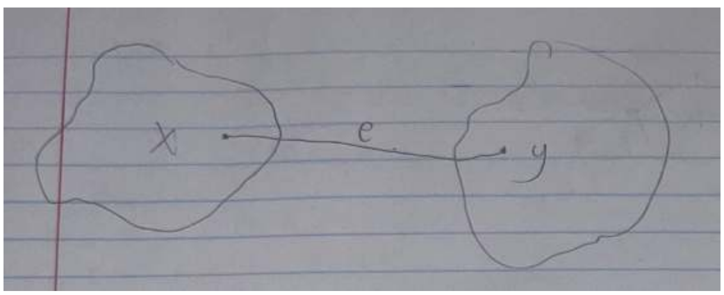
Def :
Let G be a graph and $e \in E(G)$. We write $G-e$ for the graph with vertex set $V(G)$ and edge set $E(G)-\{e\}$.

Def :
An edge e of G is called a bridge of G if $G-e$ has more components than G has. In particular, if G is connected then e is a bridge if $G-e$ is disconnected.

Lemma: If $e = xy$ is a bridge of connected graph G, then $G-e$ has exactly two components, C_x contains x, and C_y contains y.

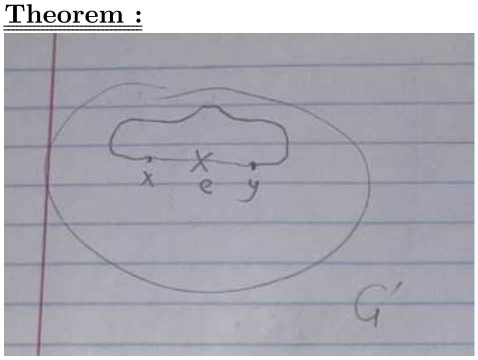
Pf.

Suppose z is not in the component C_x containing x. Since G was connected, there was a path P in G from x to z. Hence, P must contain e, say $P = x(e)y_1y_2\ldots y_kz$. Then $yy_1y_2\ldots y_kz$ is a path in $G-e$ joining y to z. Hence, $z \in C_y$.
(Clearly $G-e$ has at least 2 components by definition of bridge.) $\square$



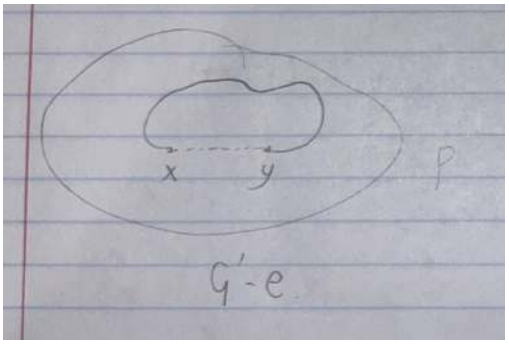
Lemma: If $e = xy$ is a bridge of a connected graph G, then $G-e$ has exactly two components G_x containing x and G_y containing y.

Theorem :



Let G be a graph. The edge e is a bridge in G if and only if e is not in any cycle of G.

Pf.



Let G' be the component at G containing e.

($\Rightarrow$)

Suppose e is in a cycle C of G. Let $e = xy$, then in $G' - e$, $C - e$ is a path from x to y. So x and y are in the same component of $G' - e$.

Hence, by the lemma, e is not a bridge of G. (hence e is not a bridge of G)

($\Leftarrow$)

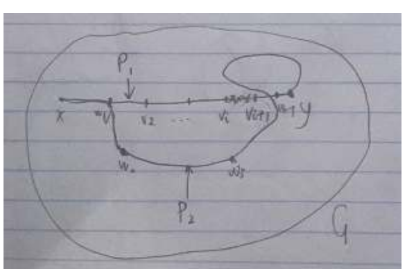
Suppose e is not a bridge of G'. Then by definition of bridge, $G' - e$ is connected. Hence, there is a path P in $G' - e$ from x to y. Then P together with $e = xy$ forms a cycle in G' containing e.

$\square$

Def :
A tree is a connected graph with no cycles.

Proposition: Let G be a tree. Then for every pair of vertices x and y, there is a unique path joining x to y in G.

Pf.



Since G is connected, there is at least one path joining x to y. Suppose that $P_1 = v_0, v_1, \ldots, v_k$ is a path $v_0 = x$ to $v_k = y$ and $P_2 = w_0, w_1, \ldots, w_l$ is another path from $w_0 = x$ to $w_l = y$.

Suppose P_1 and P_2 are not identical. Then there exists an edge, say $v_i v_{i+1}$ in P_1. (WLOG) but not in P_2.

Then we can find a walk in $G - e$, where $e = v_i v_{i+1}$, from v_i to v_{i+1}: $v_i v_{i-1} \ldots v_0(x) P_2 y v_{k-1} \ldots v_{i+1}$.

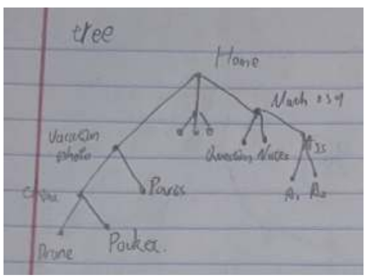
Hence, by our earlier theorem, there exist a path P from v_i to v_{i+1} in $G - e$.

Then P together with e is a cycle in G containing e, a contradiction. Hence, the path from x to y is unique. $\square$

iii. Tree

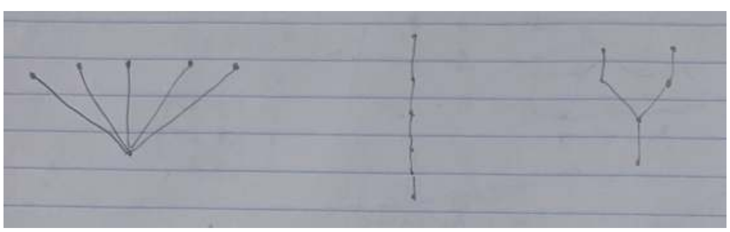
<u>Some properties of trees:</u>

- every pair of vertices joined by a unique path
- every edge is a bridge



Q: How many edges can be in a tree with P vertices?

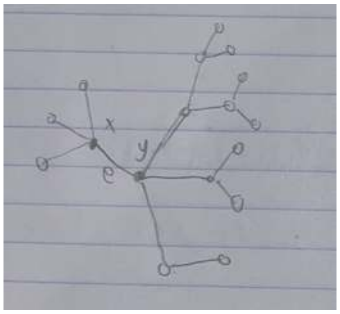
[graph: 6 vertices]



<u>Theorem :</u>
Every tree with $p \geq 1$ vertices has exactly $p - 1$ edges.

<u>*Pf.*</u>

We use induction on P.



Base case: $P = 1$. The tree is just a single vertex, so there are 0 edges.

IH: Assume $p \geq 2$ and that every tree with p' vertices, where $1 \leq p' < p$, has exactly $p' - 1$ edges.

Let T be a tree with p vertices. Since $p \geq 2$ and T is connected, s.t. contains on edge $e = xy$.

Then e is a bridge, so $G' = T - e$ has exactly two components T_x containing x and T_y containing y.

Then T_x is connected (by definition of component) and has $p_x \geq 1$ vertices, and contain no cycles. Hence, T_x is a tree.

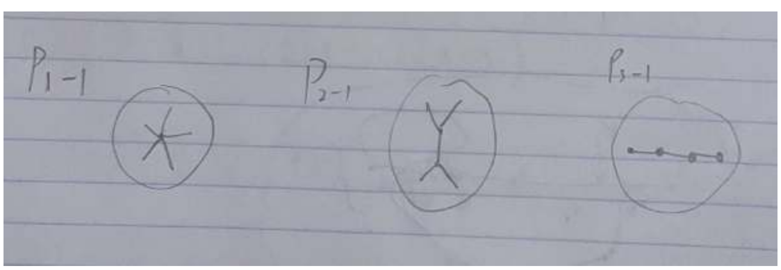
Similarly, T_y is a tree with $p_y \geq 1$ vertices.

Since $p_x + p_y = p$, and $p_x \geq 1, p_y \geq 1$, we see $p_x < p$ and $p_y < p$.

Hence, by IH, Y_x has exactly $p_x - 1$ edges and T_y has exactly $p_y - 1$ edges.

So $|E(T)| = |E(T_x)| + |E(T_y)| + \underbrace{1}_{e} = p_x - 1 + p_y - 1 + 1 = p - 1$. $\square$

Def :
A forest is a graph with no cycles. (Hence each component of a forest is a tree.)



Corollary: A forest with $p \geq 1$ vertices and $k \geq 1$ components has exactly $p - k$ edges.

Def :
A leaf in a tree is a vertex of degree 1.

Note that every tree (with at least 2 vertices) has a leaf, by our earlier theorem that every graph in which all degrees are ≥ 2 contains a cycle.

Theorem :
Let T be a tree with $p \geq 2$ vertices. For $i = 1, 2, \ldots, p$, let n_i denote the number of vertices in T with degree i. $n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots + (p-2)n_p$.

Pf.

By Handshaking Lemma, $\sum_{v \in V(T)} \deg(v) = 2|E(T)|$.

By our previous theorem $|E(T)| = p - 1$, So $n_1 + 2n_2 + 3n_3 + \cdots + pn_p = \sum_{v \in V(T)} \deg(v) = 2(p-1)$

Also, $n_1 + n_2 + n_3 + \cdots + n_p = p$.

So

$$\begin{aligned} 2p &= 2n_1 + 2n_2 + \cdots + 2n_p \\ 2(p-1) &= n_1 + 2n_2 + 3n_3 + \cdots + pn_p \\ 2 &= n_1 - n_3 - 2n_4 - 3n_5 - \cdots - (p-2)n_p \end{aligned}$$

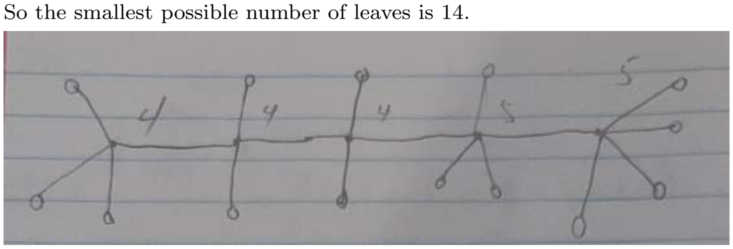
Hence, $n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots + (p-2)n_p$. □

Example:

What is the smallest possible number of leaves in a tree with 3 vertices of degree 4 and 2 vertices of degree 5?

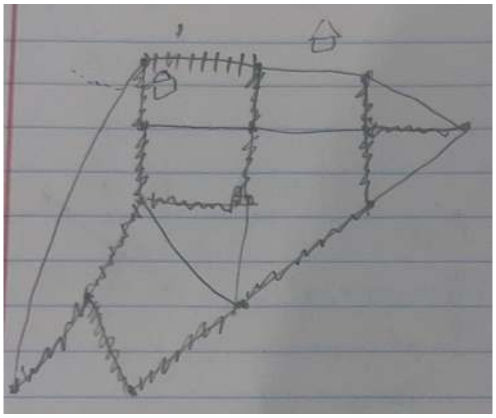
$$\begin{aligned} n_1 &= 2 + n_3 + 2n_4 + 3n_5 + \cdots + (p-2)n_p \\ &\geq 2 + 2(3) + 3(2) + \ldots \\ &\geq 14 \end{aligned}$$

So the smallest possible number of leaves is 14.



Same situation what is the largest possible number of leaves? (Exactly 14 leaves)
Answer: "∞" i.e. no upper bound.

1. Spanning trees



Def :
A spanning tree in a graph G is a subgraph of G that is a tree, and $V(T) = V(G)$.

Theorem :
A graph G has a spanning tree if and only if G is connected.
Pf.

($\Rightarrow$) If T is a spanning tree then there exists a path P_{xy} in T joining every pair x and y of vertices in $V(T)$. Then P_{xy} is a path in G joining x and y, and $V(G) = V(T)$. So G is connected.
($\Leftarrow$) Assume G is connected. If G has no cycles then it is a tree, hence its own spanning tree. Otherwise, there exists a cycle in G. Let $e \in E(C)$. Then e is not a bridge of G, so $G - e$ is connected. By repeating this, we eventually obtain a spanning tree of G. $\square$

Corollary: If G is a connected graph with p vertices and $p - 1$ edges, then G is a tree.
Pf.

Since G is connected it has a spanning tree T. $|E(T)| = p - 1 = |E(G)|$. Hence, $G = T$. $\square$

Theorem :
Let G be a connected graph and let T be a spanning tree of G. Let $e \in E(G) - E(T)$. Then

(i) The graph $T + e$ obtained by adding e to T contains exactly one cycle C_1 and C contains e. (The "fundamental" cycle for e.)

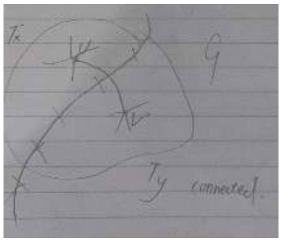
(ii) $(T + e) - f$ is a spanning tree of G for each $f \in E(C)$.

Pf.

(i) Since T is a tree, $T + e$ contains a cycle C_1. Since T is a spanning tree, $V(T) = V(G)$, so C_1 contains all vertices of G.
Let $e = xy$ Since T has no cycles, all cycles in $T + e$ contain e. There is a unique path P in T from x to y, hence $P - e$ is the unique cycle in $T + e$ that contains e.

(ii) Let $f \in E(C)$. Then $C = C_1 - e + f$. Since T is a tree, $T + e - f$ contains no cycles, so $(T + e) - f$ is a tree. Since $V(T) = V(G)$, $(T + e) - f$ is a spanning tree of G.
Since f is in a cycle in $(T + e) - f$ is connected. Since $V(T + e - f) = V(T) = V(G)$ and $|E(T + e - f)| = |E(T)| = |V(G)| - 1$. Hence, by the corollary, $T + e - f$ is a tree, and hence a spanning of G. $\square$

Theorem :
Let G be a connected graph and T a spanning tree in G.



Let $e \in E(G)$, Let $e = xy$. Then

1. $T - e$ has exactly two components, T_x containing x and T_y containing y.
2. For each edge f in the cut in G induced by $V(T_x)$, $(T - e) + f$ is a spanning tree of G.

Pf.

Since e is a bridge of T, (a) follows by our earlier theorem.
(b) Let $f = uv$ where $u \in V(T_x)$ and $v \in V(T_y)$. Every vertex in $V(T_y)$ is joined to v by a path (since T_y is connected.)
Every vertex in $V(T_x)$ is joined to v by a path in T_x to a, together with the edge e.
Hence every vertex in $V(G)$ is joined to v by a path in $(T_x \cup T_y - e) + f$.
Hence $T + e - f$ is connected. Since it has $|E(T)|$ edges, it is a spanning tree. $\square$

Finding a spanning tree start by setting

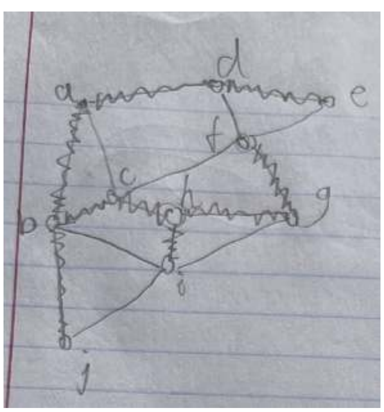
$V(T) := \{a\}$ where a is any vertex in V. $E(T) := \emptyset$

While there exists an edge $e = uw \in E(G)$ with $u \in V(T)$ and $w \notin V(T)$:

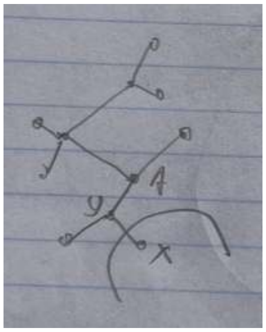
Choose such an edge uw:

- put e into $E(T)$
- put w into $V(T)$

Stop Output T, a spanning tree of G.



Lemma: Every tree is bipartite.



Pf.

By induction on P, the number of vertices in the tree T.

Base case: $P = 1$. The tree with one vertex is bipartite.

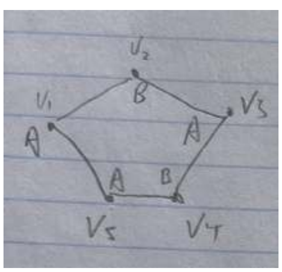
IH: Assume $P \geq 2$ and every tree with $P - 1$ vertices is bipartite.

Let T be a tree with practices. Since $p \geq 2$, T has a leaf. Then T - x is a tree with $P - 1$ vertices, which is bipartite by IH.

For a bipartition $A \cup B$ of $T - x$, WLOG the neighbor y of x in T is in A. Then $A \cup (B \cup \{x\})$ is a bipartition of T. $\square$

Question: How to characterize the bipartite graphs?

Answer: A graph is bipartite if and only if it contains no odd cycle (i.e. cycle of odd length).



Pf.

$(\Rightarrow)$

Suppose $C = v_0 v_1 \cdots v_k v_0$ is an odd cycle, i.e. k is odd and $v_1 v_k$ is an edge. If C is bipartite, then WLOG

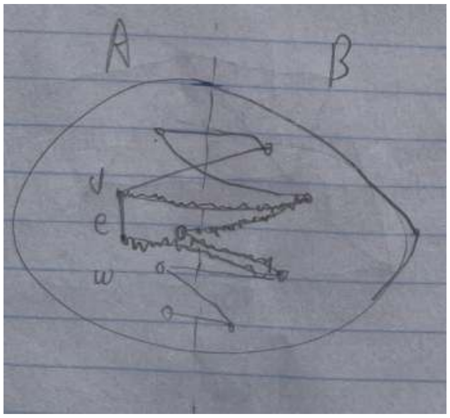
- $v_i \in A$. Then $v_i \in B$ for even i.
- $v_i \in A$. Then $v_i \in A$ for odd i.

But then $v_1 v_k$ is an edge between two vertices in the same set, which is a contradiction.

($\Leftarrow$)

Assume G is a graph that has no odd cycle.

Suppose G is not bipartite. Then it has some component G' that is not bipartite. Let T be a spanning tree of G'. Then T is bipartite, so there exists a partition $A \cup B$ of $V(T) = V(G')$ such that each edge of T has one end in A and the other in B.



Since G is not bipartite, there exists an edge of G (which is not in T) that joins two vertices in WLOG A. Call this edge $e = uw$. Then u and w are joined by a (unique) path P in T.

Since T is bipartite with parities $A \cup B$, the path $P = uv_1v_2 \cdots v_k w$ has $u \in A$ so $v_1 \in B$ and in general $v_i \in B$ if i is odd and $v_i \in A$ if i is even.

So k is odd, hence $v_k \in B$. So the path P has even length. Hence, $P \cup \{e\}$ is an odd cycle in G, contradiction. So G is bipartite, with bipartition $A \cup B$.

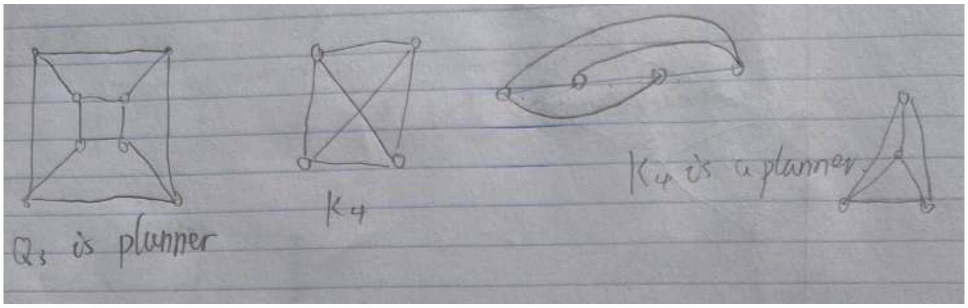
$\square$

2. Planner Graphs

Def :
A graph G is called planner if it can be drawn in the plane $\mathbb{R}^2$ such that,

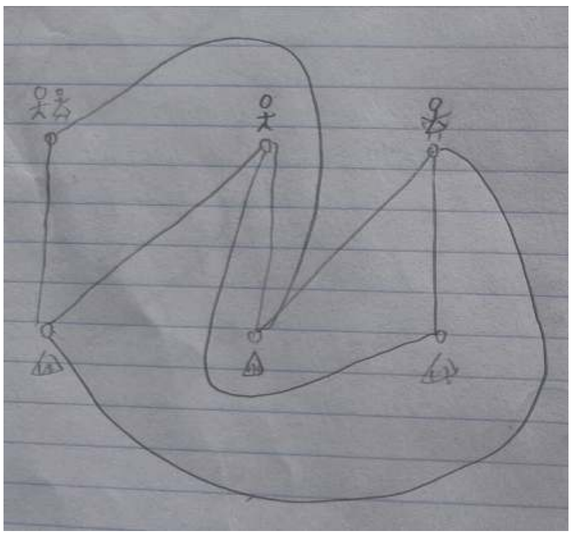
- no two edges intersect (except at their common endpoints, if it exists)
- no two vertices coincide. ("inconclusive")

Example:



Asking: Is $k_{3,3}$ planner?

Answer: No.



We will find out a concrete way to prove that a graph is not planner.

A drawing of a planner graph G that showing it is planner is called a

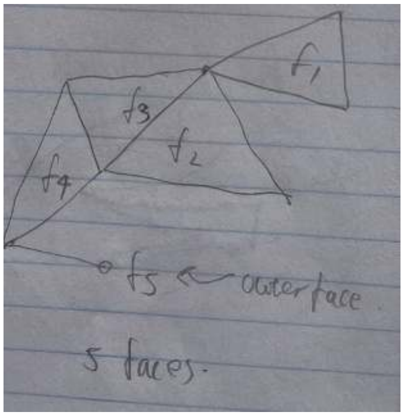
- planer embedding of G.
- planer drawing of G.
- planer map of G.

A graph is planner if and only if all its components are planner. So we will usually focus on connected planner graphs.

Def :
Let C_n be a connected planar graph, and let $\tilde{G}$ be a planar drawing of G.

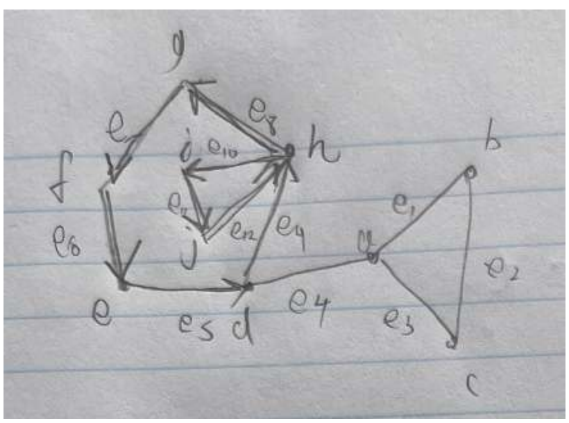
Then $\tilde{G}$ partitions the plane into connected regions called the faces of G.

One face of G is unbounded. We call it the <u>outer face</u> of G.



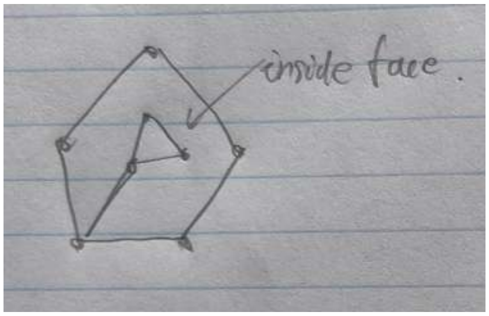
For a face f of G, the set of vertices edges on the perimeter of f is the <u>boundary</u> of f. The vertices and edges of the boundary of f are said to be <u>incident</u> to f. Two faces are said to be <u>adjacent</u> if they are both incident to the same edge.

For a face f and a vertex v_0 incident to f the sequence $v_0, v_1, \ldots, v_k, v_0$ of all the vertex and edges of the boundary of f listed in order is which they are encountered as one more around the whole perimeter of f.



Every boundary walk, vertices might to might appear several times, edges appear either once or twice.

The boundary walk of a face f is unique up to choice of beginning vertex and direction.



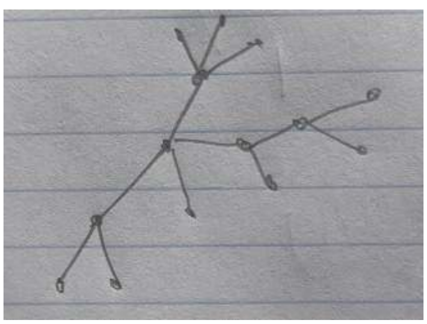
The number of edges (counted with multiplicity) in the boundary walk of f is the degree of f.

Notes:

- If edge e is in a cycle, then e is incident to two different faces. If e is a bridge then it is incident to the same face on both sides.
- If G is a tree, any planner drawing of G has one face so if $|V(G)| = p$, this face has degree $2|E(G)| = 2p - 2$.

Theorem :
Let G be a connected planner graph and $\tilde{G}$ a planner drawing of G. Let $F(\tilde{G})$ be the set of faces of $\bar{G}$.

Then
$$\sum_{f \in F(\tilde{G})} \deg(f) = 2|E(G)|$$



Pf.

Each edge e contributes 1 to the degree of the two faces incident to it, if it is not a bridge. It contributes 2 to the unique face incident to it, if it is a bridge.

Hence,
$$\sum_{f \in F(\tilde{G})} \deg(f) = 2|E(G)|$$

(”Face Shake Lemma”) □

Theorem Eular’s Formula:

For any planner drawing $\tilde{G}$ of a connected planner graph G.

$$|V(G)| - |E(G)| + |F(\tilde{G})| = 2$$

<u>*Pf.*</u>

Let $p = |V(G)|, q = |E(G)|, s = |F(\tilde{G})|$.

Fix p and use incident on q.

Base case: $p = 1$. Then $q = 0, s = 1$. $p - q + s = 2$.

IH: Assume $q \geq p$ and the statement is true for all graphs with $q = 1$ edges satisfying the condition.

Let G with parameters p, q, s be given. Since $q \geq p$, G has a cycle. Let e be an edge in a cycle. Then e is incident to two distinct faces of $\tilde{G}$. Then $G - e$ is a planner, $\tilde{G} - e$ is a planner drawing. $V|G - e| = p, |E(G - e)| = q - 1, |F(\tilde{G} - e)| = s - 1$.

Also, $G - e$ is connected since d is not a bridge of G.

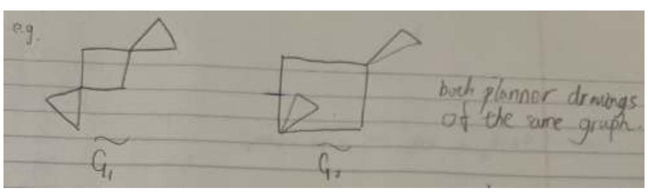
Hence, by IH $p - (q - 1) + (s - 1) = 2$.

Then $p - q + s = 2$. □

This implies that all planar embeddings of a planar graph have the same number of faces.

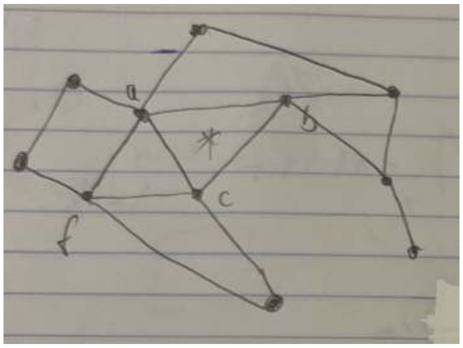
Can a planar graph have several embeddings?

Example:



Face degrees in $\tilde{G_1}$: 3,3,4,10
Face degrees in $\tilde{G_2}$: 3,3,7,7
Note:

- The set of graphs that can be drawn in the plane without crossing edges
- The set of graphs that can be drawn on the sphere without crossing edges

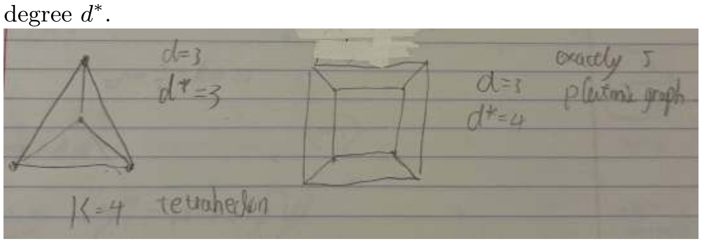


Consequence: If $\tilde{G}$ is a planar drawing of a connected planar graph G and f is a face of $\tilde{G}$, there exists a planar drawing of G in which f is the outer face.
Example:

> Find the number of faces of any planar drawings of connected planar graph G with p vertices that is 4-regular.
> HL: $g = |E(G)|$. Then $2q = \sum_{v \in V(G)} deg(v) = 4p \Rightarrow q = 2p$.
> By Euler's formula, $p - q + s = 2 \Rightarrow p - 2p + s = 2 \Rightarrow s = p - 2$.

3. Platonic Graphs

Def :
A graph is called Platonic if it is planar, all vertices have the same degree d, and all faces have the same degree d^*.



Lemma: Let G be a connected planar embedding of a planar graph with p vertices, q edges and s faces. Suppose each vertex has degree $d \geq 3$ and each face has degree $d^* \geq 3$. Then

$$(d, d^*) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}$$

Also

$$q = \frac{2dd^*}{2d \cdot 2d^* - dd^*}, p = \frac{2q}{d}, s = \frac{2q}{d^*}$$

Pf.

By HL, $2q = \sum_{v \in V(G)} deg(v) = dp \Rightarrow q = \frac{2dp}{2d} = \frac{2q}{d}$.
By "Face Handshaking Lemma", $2q = \sum_{f \in F(G)} deg(f) = d^* s \Rightarrow s = \frac{2q}{d^*}$.

By Euler's formula,

$$\begin{aligned}
p - q + s &= 2 \\
\frac{2q}{d} - q + \frac{2q}{d^*} &= 2 \\
q\left(\frac{2}{d} - 1 + \frac{2}{d^*}\right) &= 2 \\
q\left(2d^* + 2d - 2d\right) &= 2d \cdot d^* \\
2d^* - dd^* + 2d &> 0 \\
2d^* - dd^* - 2d &< 0 \\
(2 - d^*)(d - 2) &< 0
\end{aligned}$$

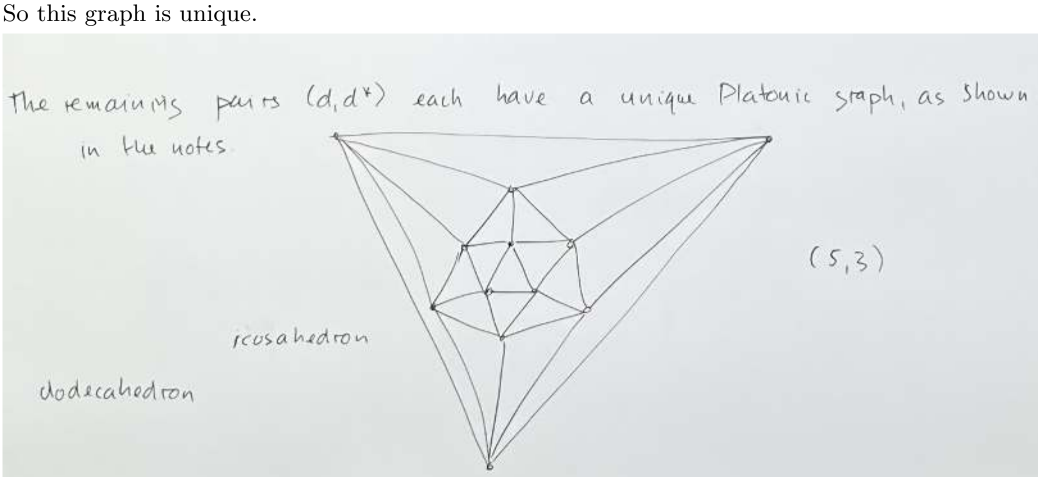
But d and d^* are integers, both ≥ 3. So the only possibilities are

- $d = 3, d^* = \{3, 4, 5\}$
- $d = 4, d^* = 3$
- $d = 5, d^* = 3$

□

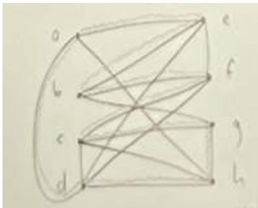
Now we'd like to show that for each possible pair of parameters (d, d^*), there is exactly one graph.

- $(d, d^*) = (3, 3)$: K_4 is an example. It has $p = 4$ vertices and is 3-regular
  K_4 is the unique graph with these parameters.
- $(d, d^*) = (4, 3)$: Then $q = \frac{2dd^*}{2d + 2d^* - dd^*} = 12$.
  Then $p = \frac{2q}{d} = 6$ and $s = \frac{2q}{d^*} = 8$.
  A graph with these parameters has a pairing of vertices $\{v, \tilde{v}\}$ such that v's only non-neighbors is $\tilde{v}$, and v's only non-neighbors is $\tilde{v}$.
  So this graph is unique.

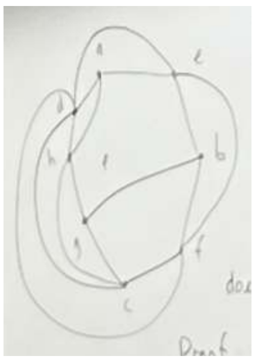


Q: How to show a graph is planner?

A: Exhibit a planar embedding.



- Look for a long cycle C in the graph.
- Embed C in the plane, as a 'circle'.
- Try to fit in the remaining vertices and edges, either inside or outside the circle.

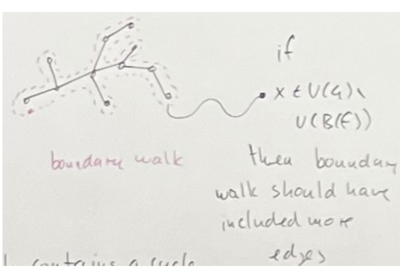


Lemma Let $\tilde{G}$ be a planner embedding of a connected planar graph G. Let f be a face of G. If the boundary $B(f)$ does not contain a cycle, then G is a tree.

Pf.

The boundary walk of f contains all vertices and edges of $B(f)$, so $B(f)$ is a connected subgraph of G. Hence, $B(f)$ is a tree since G is connected, it is not possible for vertices $x \in U(G)\backslash U(B(f))$ to be present. Hence, $G = B(f)$, which is a tree. $\square$

Contrapositive: If a connected planar graph contains a cycle, then every face of every planar embedding contains a cycle in its boundary to particular, every face has edges ≥ 3.



Theorem :
Let G be a planar graph. Then $|E(G)| \leq 3|V(G)| - 6$.

Pf.

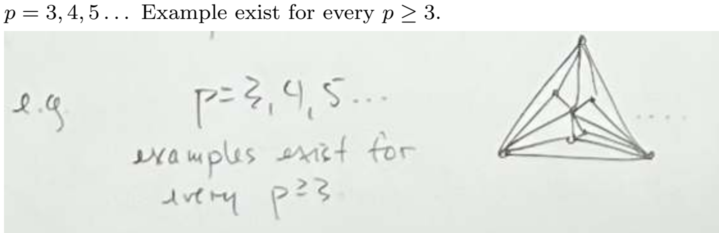
If G is not connected, we may add edges between components one at a time, so that the resulting graph is still planar. This does not change $P = |V(G)|$ and increases $q = |E(G)|$, so it suffices to prove the statement for connected planar graphs. Hence, we may apply Eular's Formula. $\square$

- If G does not contain a cycle then G is a tree. Then $q = p - 1 \leq 3p - 6$.
- If G does contain a cycle then every face of every planar embedding $\tilde{G}$ of G contains a cycle in its boundary. So $deg(f) \geq 3$ for every face of $\tilde{G}$. Let $F(\tilde{G})$ denote the set of faces. Then by Face-shake Lemma. $2q = \sum_{f \in F(\tilde{G})} deg(f) \geq 3s$, where $s = |F(\tilde{G})|$. By Euler's Formula $p - q + s = 2$ we get $s = 2 + q - p$.
  Hence, $2q \geq 3(2 + q - p)$, which implies $q \leq 3p - 6$.

Is it best possible? i.e. Do there exist planar graphs with p vertices and exactly $3p - 6$ edges?

Example:

$p = 3, 4, 5 \ldots$ Example exist for every $p \geq 3$.

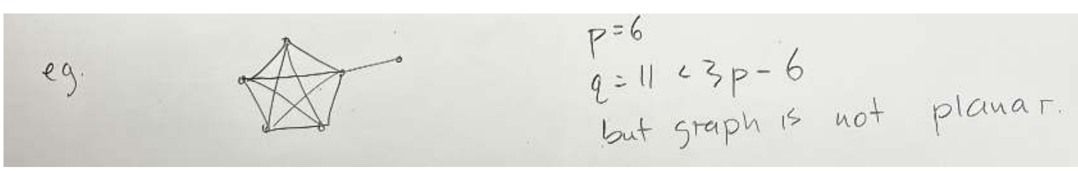


Col: K_5 is not planar.

Pf.

We have $p = |V(K_5)| = 5$, $q = |E(K_5)| = 10$, and $3p - 6 = 9 < q$. So K_5 is not planar. $\square$

Note: If $q \leq 3p - 6$ it does NOT mean graph is planar!



Theorem :
Suppose G is a planar graph in which every face has degree at least k. Then $|E(G)| \leq \frac{k}{k-2}(|V(G)| - 2)$.

Pf.

Same as previous proof until $2q = \sum_{f \in F(\tilde{G})} deg(f) \geq ks$, where $s = |F(\tilde{G})|$. By Euler's Formula:

$$\begin{aligned} 2q &\geq k(2+q-p) \\ 2q &\geq 2k + kq - kp \\ k(p-2) &\geq (k-2)q \\ q &\leq \frac{k}{k-2}(p-2) \end{aligned}$$

□

Def :
The girth of a graph is the length of its shortest cycle (infinite, or not defined, if G has no cycles).

Cor: Any planar graph G with finite girth k has at most $\frac{k}{k-2}(|V(G)|-2)$ edges.

Cor: $K_{3,3}$ is not planar.

Pf.

Since $K_{3,3}$ is bipartite, it has no 3-cycle. Its girth is 4. Then since $q = 9, p = 6$ for $K_{3,3}$, but $\frac{y-2}{2}(p-2) = 8$. So $K_{3,3}$ is not planar. □

Cor: Let G be a planar graph. Then G has a vertex of degree ≤ 5.

Pf.

Let $q = |E(G)|$ and $q = |V(G)|$. Then by our previous theorem $q \leq 3p-6$. Suppose on the contrary that all vertices of G have degree ≥ 6. Then by HL.

$$2q = \sum_{v \in V(G)} \deg(v) \geq 6p \text{ Hence } q \geq 3p, \text{ contradicting } q \leq 3p-6.$$

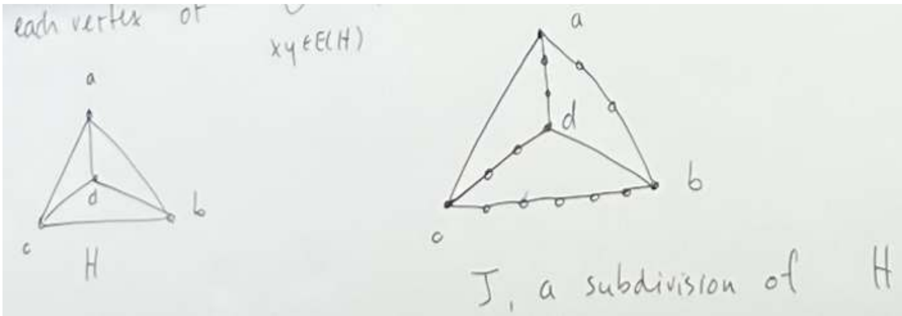
□

Q: Do there exist planar graphs such that all vertices have degree ≥ 5?

A: Yes! For example, the icosahedral.

4. Subdivisions

Def :
Let H be a graph. An (edge) subdivision of H is any graph J obtained by replacing each edge $e = xy$ of H by a path P_{xy} from x to y, of length ≥ 1, such that each vertex of $\cup_{xy \in E(H)}(V(P_{xy}) \backslash \{x, y\})$ has degree 2 in J.



A graph is planar if and only if every subdivision of it is planar.

Q: How to tell if a graph is NOT planar?

Fact: If a graph G contains a subdivision of k_5 or $k_{3,3}$, then G is not planar.

Pf.

Any subdivision of k_5 or $k_{3,3}$ is not planar, so G is not planar. $\square$

Theorem Kuratowski's theorem:

A graph is planar if and only if it does not contain a subdivision of k_5 or $k_{3,3}$.

Pf.

($\Rightarrow$): as fact above.

($\Leftarrow$): Use as fact, beyond the scope of this course (CO 342) $\square$

A: Exhibit a subgraph of G that is a subdivision of k_5 or $k_{3,3}$.

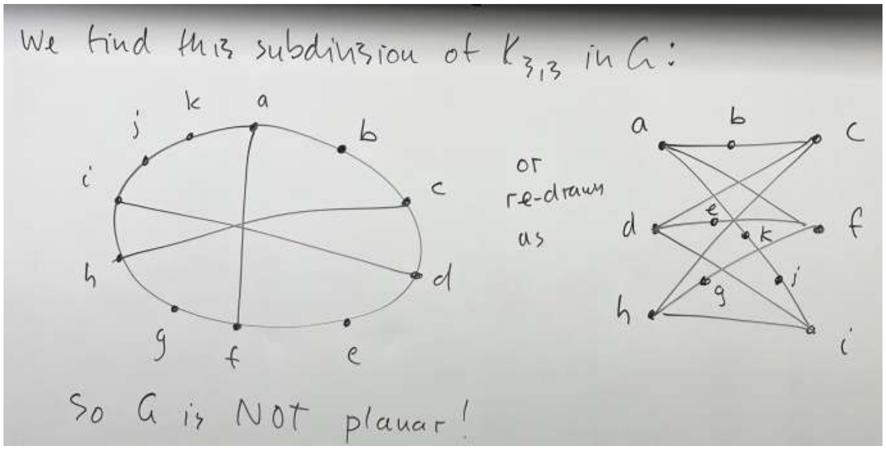
Given a graph, how to determine whether it is planar?

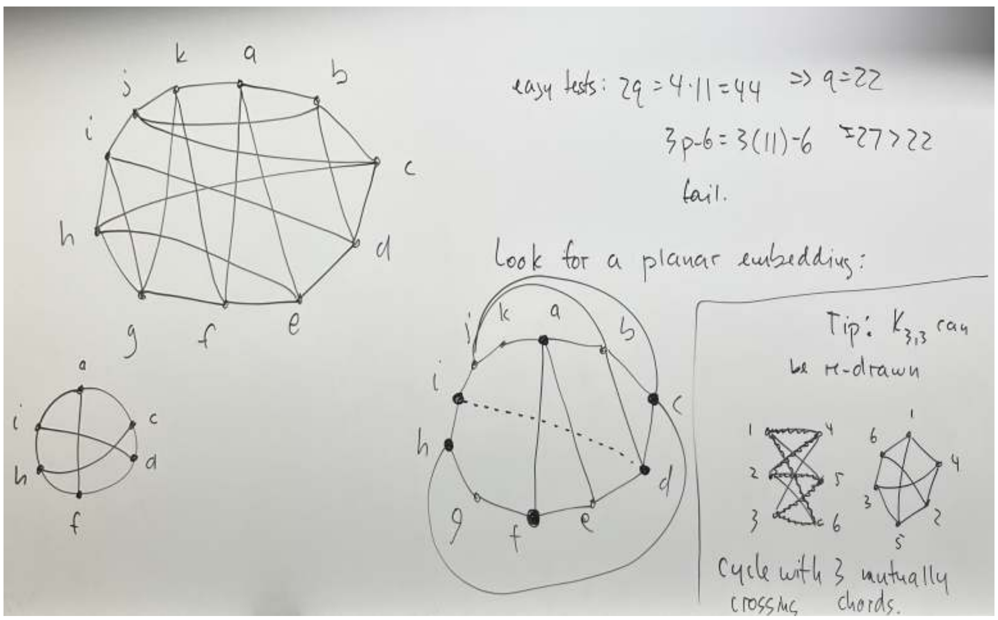
Following are some criteria to check (NOT planar if any of the following is true):

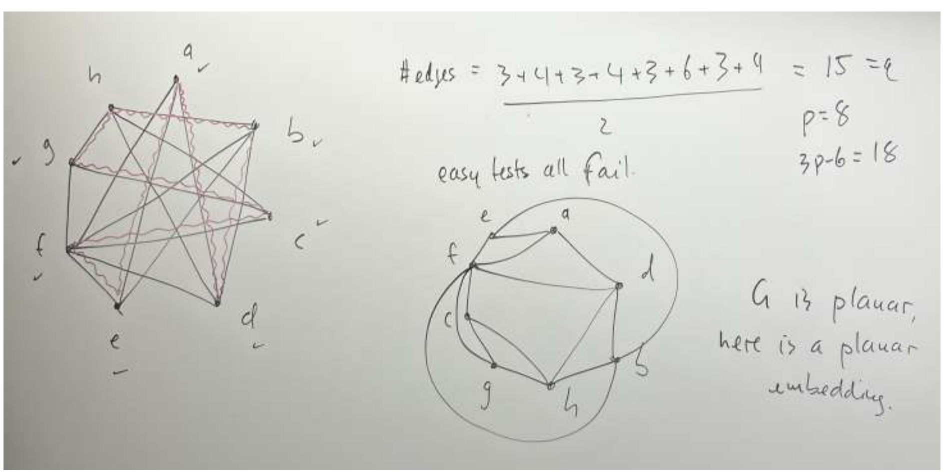
- Do all vertices of G have degree ≥ 6?
- Is $q > 3p - 6$?
- Is $q > \frac{K}{k-2}(p-2)$ where k is the girth of G?

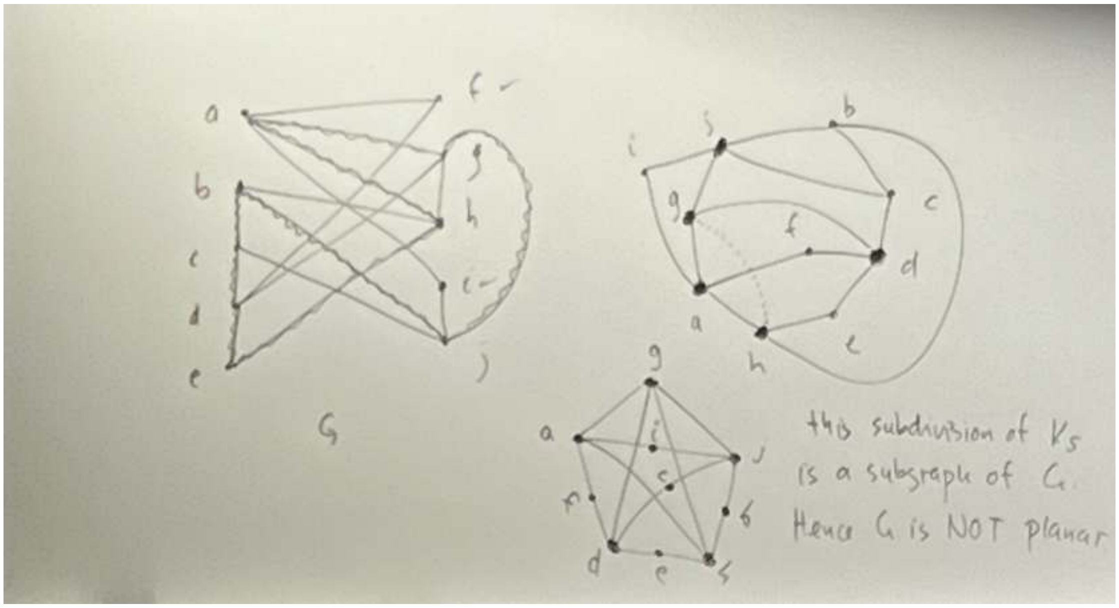
Otherwise:

- Can we find a planar embedding of G? $\Rightarrow$ G is planar.
- Can we find a subdivision of k_5 or $k_{3,3}$ in G? $\Rightarrow$ G is not planar.


We find this subdivision of $K_{3,3}$ in G:
or re-drawn as
So G is NOT planar!


easy tests: 2q = 4·11 = 44 => q = 22
3p-6 = 3(11)-6 = 27 > 22
fail.
Look for a planar embedding:
Tip: $K_{3,3}$ can be re-drawn
cycle with 3 mutually crossing chords.

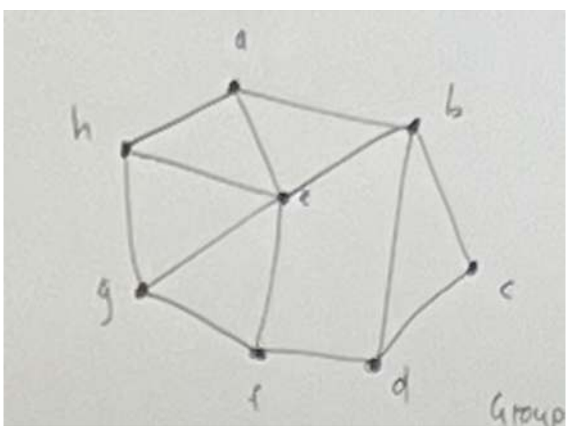

#edges = (3+4+3+4+3+6+3+4)/2 = 15 = q
p=8
3p-6 = 18
easy tests all fail.
G is planar, here is a planar embedding.



iv. Graph Coloring

Find a partition of $V(G)$ into groups, so that no two vertices in the same group are adjacent.

Aim: minimize the number of groups needed.



- Groups 1: $\{e, d\}$
- Groups 2: $\{b, f, h\}$
- Groups 3: $\{a, c, g\}$

Def :

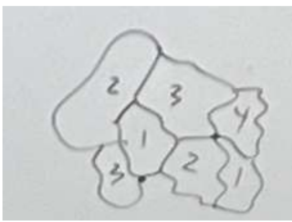
Let G be a graph. A coloring of G is a function $f : V(G) \to \mathbb{N}$ such that $f(x) \neq f(y)$ for every edge $xy \in E(G)$.

If G has p vertices then there exists a coloring $f : (G) \to \{1, 2, ..., p\}$. A coloring $f : V(G) \to \{1, 2, ..., k\}$ is called a k-coloring of G, and G is said to be k-coloring if there exists a k-coloring of G.

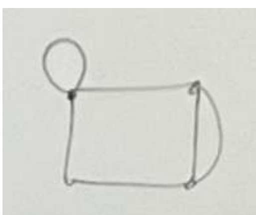
Note that if G is k-colorable it is also l-colorable for all $f \geq k$.

Famous conjecture theorem (The four-color theorem): every geographical map is 4-colorable.

i.e. the countries can be colored with 4 colors so that no two countries sharing a border have the same color.



Recall that if we wish to consider loops and multiple edges, we are looking at multigraphs.



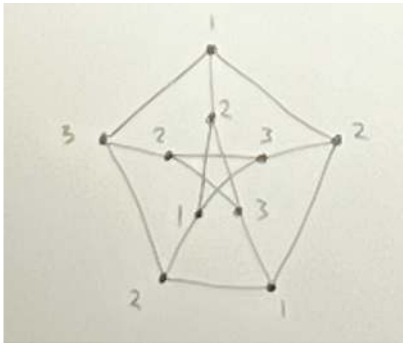
A multi-graph with a loop is not k-colorable for any k loopless. A multigraph with multiple edges is k-colorable if and only if the graph with all but our copy of each multiple edge removed is k-colorable.

Def :
Let G be a connected planar embedding. The planar dual G^* of G is the planar multigraph defined as follows:

- $V(G^*) = F(G)$, (with the vertex of G^* drawn inside each face of G)
- G^* has an edge e^* for each $e \in E(G)$, that joins the vertices corresponding to the two faces incident to e.

Some properties of the planar dual:

- $G^{**} = G$
- $d_{G^*}(f) = d_G(f)$ where v corresponds to face f.
- A (vertex) coloring of G^* is a coloring of the faces of G, so that no two faces incident to the same edge have the same color. This is a "geographical map" coloring of G.



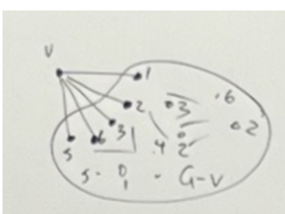
- 1 - colorable means no edges
- 2 - colorable means no odd cycles
- Every graph with prentices is p-colorable
- K_p is colorable but not l-colorable for any $l = p$

Theorem 6-color Theorem:
Every planar graph is 6-colorable.

Pf.

By induction on $P = |V(G)|.$, where G is a planar graph if $p \leq 6$ then G is 6-colorable.

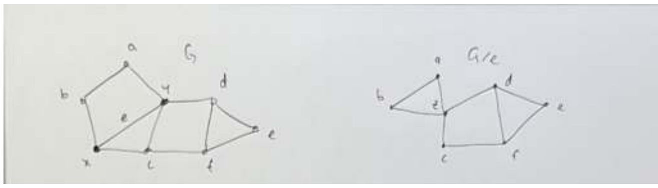
IH: Assume $P \geq 7$ and every planar graph with fewer than p vertices is 6-colorable.



Let G be a planar graph with p vertices. By an earlier theorem we know that G has a vertex v of edges ≤ 5. Then the graph $G - v$ obtained by removing v from G is planar, and has $p - 1$ vertices. Hence, by IH we find $G - v$ is 6-colorable. Let f be a coloring of $G - v$ with the color set $\{1, 2, 3, 4, 5, 6\}$.

Since v has ≤ 5 neighbors in $G - v$, there exists $c \in \{1, 2 \ldots, 6\}$ that is not used by f on the neighborhood of v.

Then we can extend f to a coloring of G by setting $f(v) = c$. Thus, G is 6-colorable. $\square$



Def :
Let G be a graph and let $e = xy$ be an edge of G. The graph $G\backslash e$ obtained by contracting e has vertex set $(V(G)\backslash\{x, y\}) \cup \{z\}$, where z is a new vertex.

The edge set is $\{uv \notin E(G) : \{u, v\} \cap \{x, y\} \neq \emptyset\} \cup \{uz : u \notin \{x, y, z\}, ux \in E(G) \text{ or } uy \in E(G)\}$.

If G is planar then $G\backslash e$ is planar.

Theorem 5-color Theorem:
Every planar graph is 5-colorable.

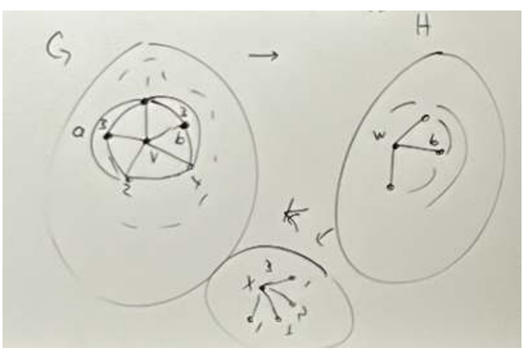
Pf.

By induction $P = |V(G)|$. If $P \leq 5$ then G is 5-colorable.

IH: Assume $p \geq 6$, and that every planar graph with fewer than P vertices is 5-colorable.

Let G be a planar graph with P vertices. By our earlier theorem, we know G has a vertex v of degree ≤ 5. If $deg(v) \leq 4$ then in the same way as in the 6-colour theorem, we can color G with 5 colors.

Then there exist neighbors a and b of v that are not joined by an edge in G. Otherwise, we would set a subgraph of G isomorphic to K_5, which is not planar.



- Contract the edge va to set $H = G\backslash va$. Then H is planar. Call the new vertex w. Then $wb \in E(H)$.
- Contract the edge wb is H to get $K = H\backslash wb$, and call the new vertex x. Then K is planar and has $p - 2$ vertices.

Hence, by IH, K is 5-colorable. Let f be a coloring of K with colors $\{1, 2, 3, 4, 5\}$. Form a coloring f' of $G - v$ by setting $f'(y) = \begin{cases} f(y) & \text{if } y \notin a, b, v \\ f(x) & \text{if } y \in a, b \end{cases}$

This is valid coloring of $G - v$ because

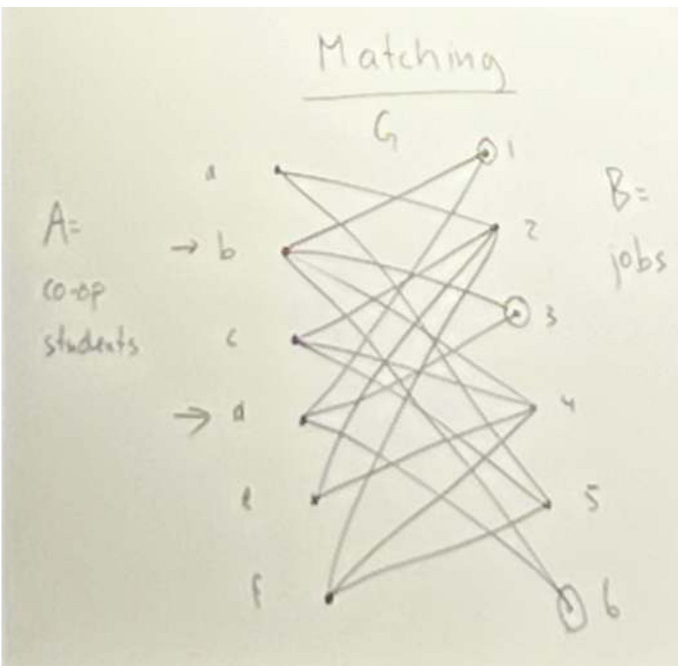
- $ab \notin E(G - v)$ so $f'(a) = f'(b)$ is valid.
- Every vertex u that is adjacent to a or b in $G - v$ is also adjacent to x in K. So since f is a coloring of K, we set $f'(u) = f(a)$ and $f'(x) \neq f'(b)$.

Hence, in $G - v$, at most 4 distinct colors are used by f' on the neighbors of v (since a and b are setting the same color).

Therefore, we say extend f' to G by setting $f'(v) = c$ to get a 5-coloring of G.

(Note: So some $c \in \{1, 2, 3, 4, 5\}$ is not used by f' on the neighbors of v) $\square$

v. Matching



Edge sj indicates that student s applied to job j.

We want to find a set M of b edges in G, such that no two edges in M share a vertex

(So this will imply each student is matched to a job, and vice versa.)

Def :
For a set S of vertex in a graph G, we write $N(S) = \{x \in V(G), (x, y) \in E(G) \text{ for some } y \in S\}$

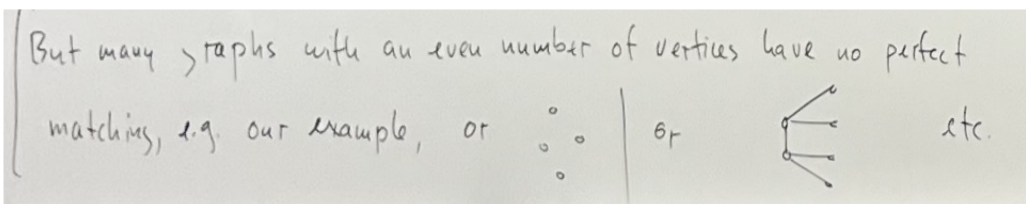
We see that $N(\{1,3,6\}) = \{b,d\}$. So we cannot find 3 edges that match 1,3,6 to distinct students. Here we see $\{1,3,6\}$ is a subset S, of B such that $|N(S)| < |S|$. So G has no perfect matching.

Def :
In any graph G, a set H of edges, no two of which share a vertex, is called a matching in G. We say a vertex x of G is saturated by M if x is incident to some edge of M. If M saturates every vertex of G we say M is a perfect matching.

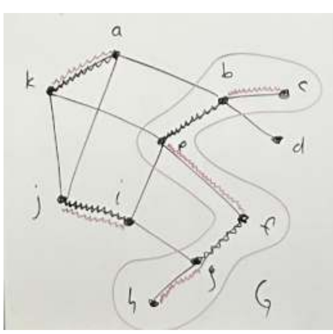
(Note: that if a perfect matching exists in G then $|V(G)|$ is even.)

But many graphs with an even number of vertices have no perfect matching.

Example:



Usually we are interested in finding a matching in G that is as large as possible, i.e. one of maximum size. This is called a maximum matching.
Example:

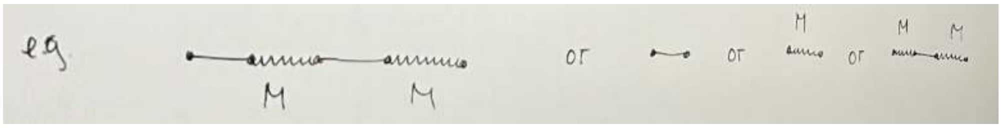


Note the path $h \to \underbrace{g \to f}_{M_1} \to \underbrace{e \to b}_{M_1} \to c$ contains 2 edges of M,
and it alternates between an edge not in M, and an edge in M_1. Also, its two endpoints h and c are not saturated by M_1.

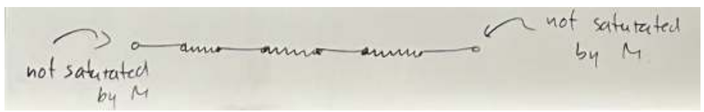
Such a path is called a (M_1)-alternating path in G. Whenever we have an (M_1)-alternating path in G, we can "switch" edges on P as we did here to crate a bigger matching in G.

Def :

Let M be a matching in a graph G. An alternating path with respect to M (or an M-alternating path) is a path in G.



An alternating path of length ≥ 1 that begins and ends with a vertex that is not saturated by M is called a (M)-augmenting path.



Note that any N-augmenting path is of odd length, and its first and last edges are not in M. Hence, it has fewer M-edges than non-M-edges.

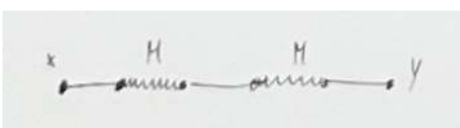
Q: How to tell whether a given matching in a graph G is a maximum matching?

Theorem :

Let G be a graph and let M be a matching in G. If there exists an $M - augmenting$ path P in G, then M is not a maximum matching.

Pf.

Recall that P being an $M - augmenting$ path means

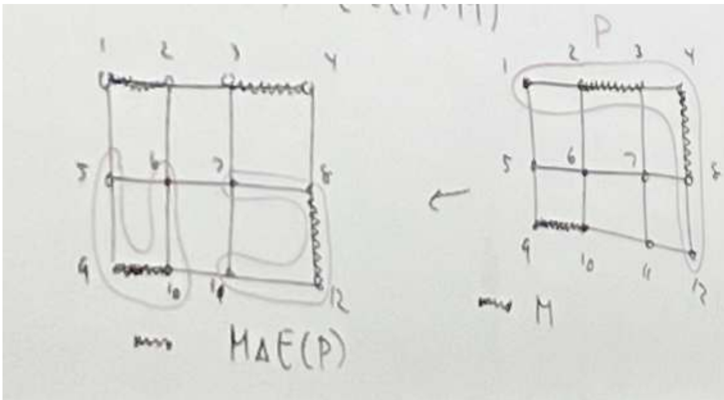


- P has length at least 1
- P is an $M - alternating$ path
- the first vertex and the last vertex of P are both unsaturated by M

Then

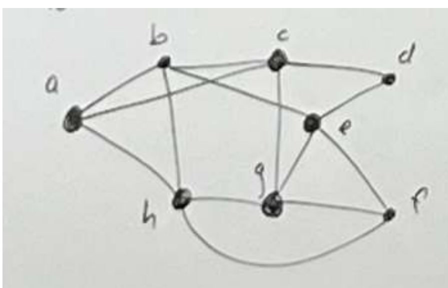
$$M \Delta E(P) = M \backslash (M \cap E(P)) \cup (E(P) \backslash M)$$

is a matching in G since P has more non-matching edges than edges in M, we find $|M \Delta E(P)| > |M|$.



Hence, M was not a maximum matching $\square$

Def :
Let G be a graph A cover of G is a set $C \subseteq V(G)$ of vertices in G with the property that every edge of G is incident to at least one vertex in C.



Lemma: Let G be a graph, let M be a matching in G and let C be a cover in G. Then $|M| \leq |C|$.

Pf.

Every edge of M must be incident to some vertex of C. Since no two edges of M share a vertex these elements of C are all distinct.

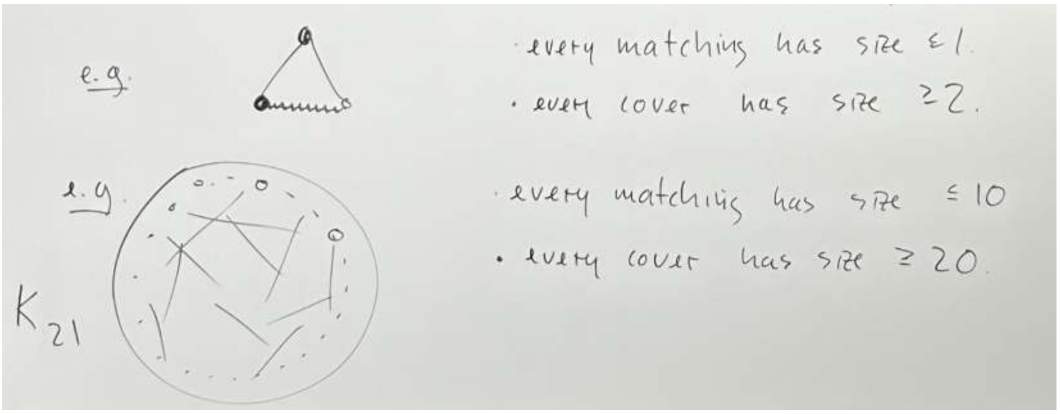
Hence, $|M| \leq |C|$. $\square$

Lemma: Let G be a graph, let M be a matching in G and let C be a cover of G. If $|M| = |C|$, then M is a maximum matching in G and C is a minimum cover of G.

Pf.

Suppose M and C exist such that $|M| = |C|$. Let M^* be a maximum matching in G. Then by the previous lemma $|M^*| \leq |C| = |M|$. Hence, $|M^*| \leq |M|$, i.e. M is a maximum matching. Similarly, let C^* be a minimum cover of G. Then by the previous lemma $|C^*| \geq |M| = |C|$. Hence, $|C| = |C^*|$, i.e. C is a minimum cover. $\square$

Note: Some graphs will NOT have a matching M and a cover C of the same size. So this test for maximum matching won't work in such graphs.

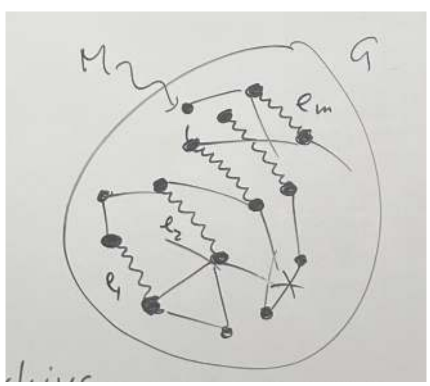


We'll prove König's Theorem, If G is bipartite, then the maximum size of a matching in G is equal to the maximum size of a cover in G. This test will then be our "stopping rule" in our algorithm for finding a maximum matching in a bipartite graph.

Corollary: In every graph, the maximum size of a matching is at least one half of the minimum size of a cover.

Pf.

Let M be a maximum matching in G. Then the set $C = \{v : v \text{ is M-saturated}\}$ is a cover of G, since if an edge xy is such that $x \notin C$ and $u \notin C$ then $M \cup \{xy\}$ is a matching larger than M. Since $|C| = 2|M|$, the conjecture is proved. $\square$



Matching M: set of disjoint edges in a graph G

Cover C: set of vertices such that all edges is incident to a vertex in C

We saw for every G, every M, every C, $|C| \geq |M|$.

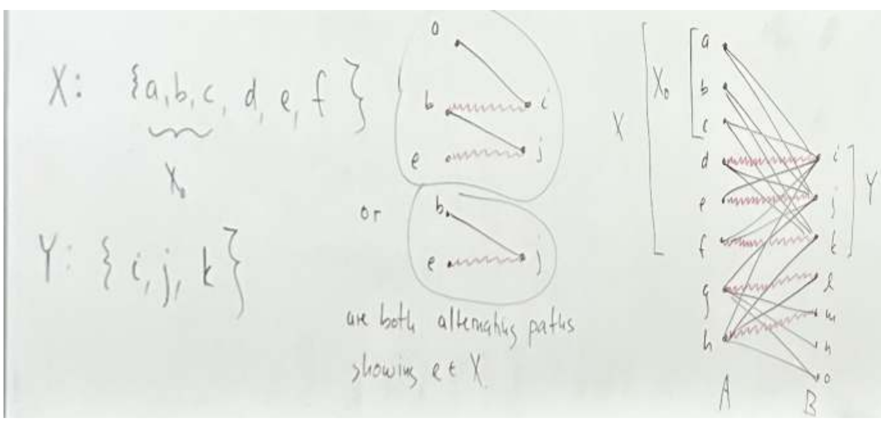
Also, to prove König's theorem. If G is bipartite then max size of a matching in $G =$ min size of a cover in G.

Let G be a bipartite graph with vertices A and B. Let M be a matching in G. [Is M a maximum matching?]

Define X_0 to be the set of all vertices in A that are not saturated by M.

[Note: if $x_0 = 4$ then M is a maximum matching, since it saturated A.]

Define the set $X \subseteq A$ to be the set of all $v \in A$ for which there exists an M alternating path $P(v)$ that starts at x_0 and ends at a vertex in B.



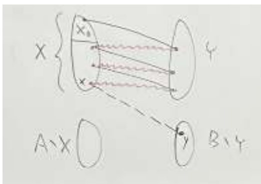
Note:

- $x_0 \subseteq X$ since for each $t \in x_0$ the path $D(x)$ of length 0 is an $M - alt$ path.
- For $v \in X$ there could be several M-alternating paths $D(v)$
- For any v and any $P(v)$, the last edge of $P(v)$ is in M.
- For $v \in Y$ there could be several $P(v)$

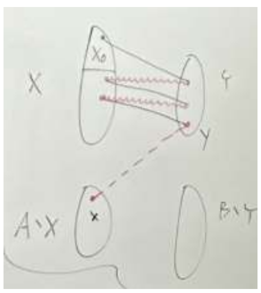
- For any $v \in Y$ and any $P(v)$, the last edge of $P(v)$ is NOT in M.

Define $Y \subseteq B$ to be the set of all $v \in B$ such that there exists an M-alternating oath that starts with a vertex of x_0 and ends with v.

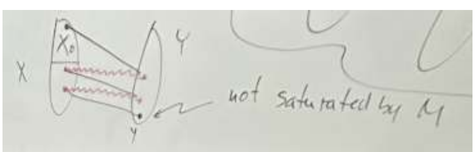
- <u>Claim 1:</u> With these definitions, there is no edge xy of G with $x \in X$ and $y \in B \backslash Y$. Otherwise, we should have put y into Y.



- <u>Claim 2:</u> With these definitions, there is no edge $xy \in M$ with $y \in Y$ and $x \in A \backslash X$. Otherwise, we should have put x into X.



- <u>Claim 3:</u> If there exists $y \in Y$ that is not saturated by M, then M is NOT a maximum matching. Then the path $P(y)$ is an M-augmenting path, so M is not maximum.

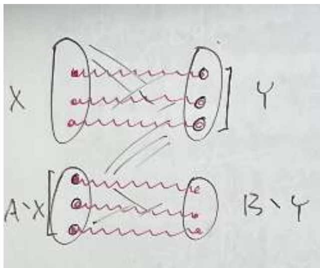


Theorem König's Theorem::
Let G be a bipartite graph and let M be a maximum matching in G. Then there exists a cover C in G with $|C| = |M|$.

Pf.

We showed already that every cover must have size at least $|M|$, so we just need to show some cover C with size $\leq |M|$. Define the sets X and Y as previously described.

By definition, $A \backslash X$ contains no vertex unsaturated by M. So there are $|A \backslash X|$ edges of M incident to $A \backslash X$.



By Claim3, every vertex in Y is saturated by M. So there are $|Y|$ edges of M incident to Y.

Hence, $|M| + |A\backslash X| = |Y \cup (A\backslash X)|$ is a cover of G.

Hence, $|C| = |M|$ as desired. $\square$

1. Bipartite Graph Matching Algorithm

<u>Outline</u>:

- Start with a bipartite graph G and a watching M in G.
- Start constructing the sets X and Y in G.
- If we find a vertex $y \in Y$ that is unsaturated by M, we'll also have an $M-augmenting$, path P that ends at y. Then we replace M by the matching. $M\Delta E(P)$ obtained by "switching" on P. This is a bigger matching start over with this bigger matching.
- If we complete the construction of X and Y without finding $y \in Y$ that is unsaturated by M, we stop then M is a maximum matching, and $C = Y \cup (A - x)$ is a cover with $|C| = |M|$.

<u>Formal description</u>:

INPUT: Bipartite graph G, vertex classes A and B.

OUTPUT: A maximum watching M, and a minimum cover C.

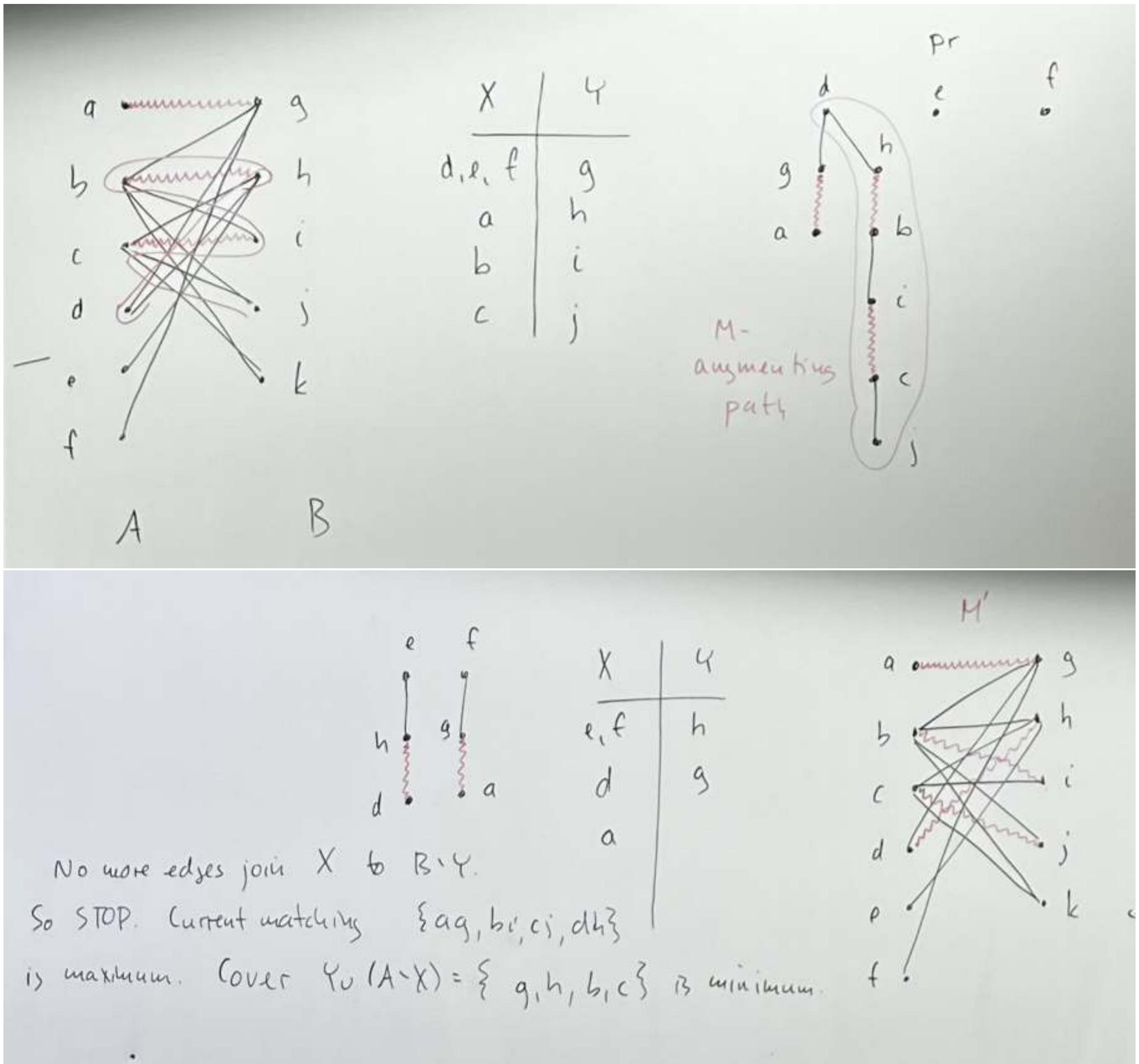
Let M be a matching in G.

1. Set $\hat{x} = \{v \in A :$ v is not saturated by M and $\hat{y} = \emptyset\}$.
2. If there is no edge joining $\hat{x}$ to $B\backslash\hat{y}$, then STOP. OUTPUT $M_0 := M$ and $C_0 := \hat{Y} \cup (A\backslash\hat{X})$.
3. Else choose $v \in B\backslash\hat{Y}$ where there exists $u \in \hat{x}$ with $uv \in E(G)$. Add v to $\hat{Y}$ and set $pr(v) := u$.

   If v is unsaturated by M then pr function indicates an M-augmenting path P that ends at v. Switch on P to get the matching $M' = M\Delta E(P)$, which is bigger than M. Replace M by M' and go to step 1.

4. If v is saturated by M, say $vz \in M$, then add z to $\hat{X}$ and set $pr(z) := v$. Go to step 2.

Example:

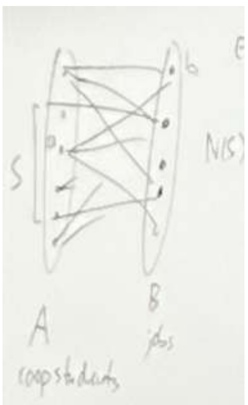


<u>Def</u> :
In a graph G, for a set $S \subseteq V(G)$, the <u>neighborhood</u> $N(S) = \{y \in V(G) \mid xy \in E(G) \text{ for some } x \in S\}$.

In a bipartite graph G with vertex classes A and B, if $S \subseteq A$, then $N(S) \subseteq B$, and vice versa.

(Edx mean "a applied to b")

If $S \subseteq A$ is such that $|N(S)| < |S|$, then there is no matching in this bipartite graph that satisfies A.



Theorem Hall's Theorem:
Let G be a bipartite graph with vertex classes A and B. Then G has a matching saturating A if and only if

1. $|N(D)| \geq |D|$ for every $D \subseteq A$.

Pf.

($\Rightarrow$)

Suppose M is a matching in G that saturates A. Let $D \subseteq A$. Then the matching edges incident to D have $|D|$ distinct end points in B. Since these are all in $N(D)$, we find $|N(D)| \geq |D|$.

($\Leftarrow$)

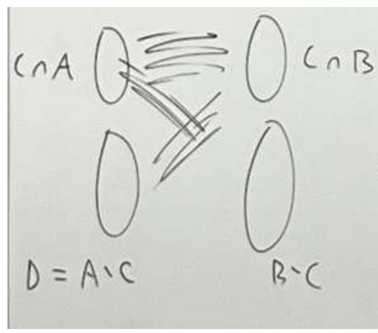
Suppose $(*)$ holds. Suppose on the contrary that G has no matching saturating A. Then the maximum size of a matching in G is $\leq |A| - 1$.

Therefore by König's Theorem, there exists a cover C in G of size $\subseteq |A| - 1$ Consider $D = A - C$. (Note

$D \neq \emptyset$). Since C is a cover. $N(D) \subseteq C \cap B$. So,

$$\begin{aligned} |N(D)| &\leq |C \cap B| \\ &= |C| - |C \cap A| \\ &= |C| - |A \backslash D| \\ &= |C| - |A| + |D| \\ &= |C| - |A \backslash C| \\ &\leq |D| - 1 \quad (|C| \leq |A| - 1) \end{aligned}$$

This contradicts $(*)$. Hence, G has a matching saturating A.



$\square$

This proof tells us that if there is no matching saturating A then the set $D = A\backslash C$ (where C is a minimum cover) certifies that $(*)$ fails. From the bipartite matching algorithm, a minimum cover is given by $C = Y \cup (A\backslash X)$. So $D = A\backslash C = X$.in our construction of sets X and Y.

Col: A bipartite graph G with vertex classes A and B has a perfect matching if and only if $|A| = |B|$ $(*)|N(D)| \geq |D|$ for all $D \subseteq A$.

Col: Every regular bipartite graph (with degree ≥ 1) has a perfect matching.

Pf.

Let $k \geq 1$ and suppose G is a k-regular bipartite graph.

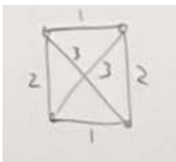
Check $|A| = |B|$: Since G is bipartite, $|E(G)| = \sum_{a \in A} deg(a) = \sum_{b \in B} deg(b)$. $k|A| = k|B| \Rightarrow |A| = |B|$.

To check $(*)$, consider $D \subseteq A$. Let $E(D, N(D))$ be the set of edges incident to D. Then $|E(D, N(D))| = \sum_{a \in D} deg(a) = k|D|$. But also $|E(D, N(D))| = \sum_{b \in N(D)} deg(b) \leq k|N(D)|$. Hence, we set $k|D| = |E(D, N(D))| \leq k|N(D)|$. So $|D| \leq |N(D)|$. Hence, $(*)$ holds, so G has a perfect matching by Hall's Theorem. $\square$

vi. Edge Coloring

Def :
An edge coloring of a graph G us a function $\beta : E(G) \to \{1, 2, 3, \ldots\}$ such that if edges e and f share a vertex that $\beta(e) \neq \beta(f)$.

In other words, β gives a partition of $E(G)$ into matching. We say G is k-edge-colorable if an edge coloring $\beta : E(G) \to \{1, 2, 3, \ldots, k\}$ exists.



Note that a vertex of degree d in G must see d distinct colors on its incident edges.

Theorem :
If G is a bipartite graph that is k-regular then G has a k-edge-coloring.

Pf.

We use induction a k. If $k = 1$, the graph is a perfect matching, so it has a 1-edge coloring. Assume $k \geq 2$ and

IH: Every (k-1)-regular bipartite graph has a (k-1)-edge-coloring.

Let G be "k-regular". By the previous result, G has a perfect matching M. The graph $G - M$ is $(k-1)$-regular and hence has a $(k-1)$-edge-coloring by IH.

Give the edges in M color k to set a $k - edge - coloring$ of G. □