

CO 342

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Note**Some Useful Terms**

- adjacent
- incident
- neighbor
- neighborhood
- degree
- complete graph ("clique")
- bipartite graph
- k-regular
- subgraph
- path
- cycle
- connected graph
- components
- tree
- planar graph
- subdivision
- face of a planar graph

Connectivity & Planar Graphs

1.1 Cut Vertices and k-Connectivity

Let G be connected, $x \in V(G)$.

Definition 1.1. x is a **cutvertex** if $G - x$ is disconnected.

For $W \subseteq V(G)$, $G - W$ is the graph obtained from G by deleting W .

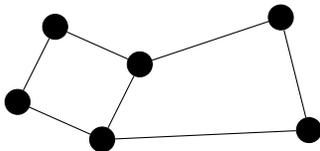
Definition 1.2. $W \subseteq V(G)$ is a vertex cut of the connected graph G if $G - W$ is disconnected.

Exercise

A connected graph that isn't complete has a vertex cut.

Definition 1.3. **K-connectivity:** Let G be a connected graph, $k \geq 1$, we say G is k -connected if:

1. $|V(G)| \geq k + 1$
2. G has no vertex cut of size $\leq k - 1$



Definition 1.4. The minimal degree $\delta(G)$ is $\min_{d(v):v \in V(G)}$, where $d(v)$ is the degree of vertex v .

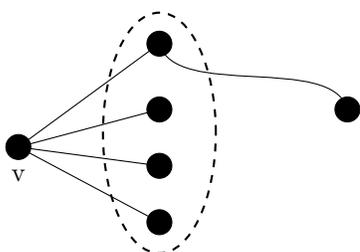
Lemma 1.1

If G is k -connected, then $k \leq \delta(G)$.

Note

$N(v)$ is the neighborhood of v .

Proof

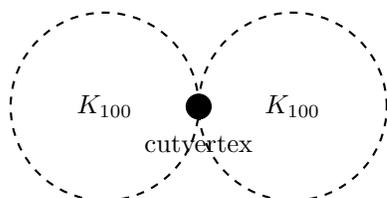


- If $|V(G)| \geq \delta(G) + 2$, then $N(v)$ is a vertex cut $|N(v)| = \delta(G) \geq k$.
- If $|V(G)| \leq \delta(G) + 1$, $k + 1 \leq |V(G)| \leq \delta(G) + 1 \implies k \leq \delta(G)$.

□

Warning

The converse is false



$\delta(G) = 99$, not 2-connected.

Lemma 1.2

Let G be a graph on n vertices, let $1 \leq k \leq n - 1$. If $\delta(G) \geq \frac{n+k-2}{2}$, then G is k -connected.

Proof

1. $|V(G)| \geq k + 1$
2. Suppose for contradiction, G has a vertex cut $W \subseteq V(G)$, $|W| \leq k - 1$.

Let H be the smallest component of $G - W$.

$$|H| \leq \frac{n - |W|}{2}$$

Let $v \in H$, $d(v) \leq |W| + (|H| - 1)$

$$\begin{aligned} \delta(G) &\leq d(v) \leq |W| + (|H| - 1) \\ &\leq |W| + \frac{n - |W|}{2} - 1 \\ &= \frac{n}{2} + \frac{|W|}{2} - 1 \\ &\leq \frac{n + (k - 1) - 2}{2} \\ &= \frac{n + k - 3}{2} \\ &< \frac{n + k - 2}{2} \\ &\leq \delta(G) \text{ (contradiction)} \end{aligned}$$

□

Note

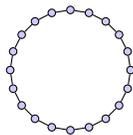
G is 1-connected v.s. G is connected

same if $|V(G)| \geq 2$, otherwise different

Recall

- $\delta(G)$: the minimum degree of G .
- If G is k -connected then $\delta(G) \geq k$.

- If $\delta(G) \geq \frac{|V(G)|+k-2}{2}$, then G is k -connected. (Converse is not true)
- G k -connected does not imply $\delta(G) \geq \frac{k+k-2}{2}, n = |V(G)|$ (e.g. C_{100} is 2-connected but $\delta(G) = 2 \ll \frac{100+2-2}{2} = 50$)

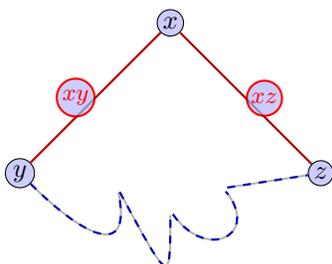


C_{20} (use C_{100} for the actual graph)

Properties of 2-connected graphs

Lemma 1.3

If G is a 2-connected graph and $xy, xz \in E(G)$, then there exists a cycle in G containing both xy and xz .



Path P from y to z in $G - x$

Proof

Since G is 2-connected, x is not a cut-vertex. So $G - x$ is connected. Hence, there exists a path P from y to z in $G - x$. Then $P \cup \{x\}$ forms a cycle in G containing xy and xz .

□

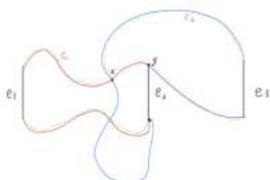
Lemma 1.4

Let G be a graph. Suppose that

- edges e_1 and e_2 lie in a common cycle in G , and
- edges e_2 and e_3 lie in a common cycle in G .

Then e_1 and e_3 lie in a common cycle in G .

Proof



Let e_1, e_2 lie in a circle c_1 and e_2, e_3 lie in a circle c_2 .

Let x be the first vertex of c_1 reached by walking along c_2 starting any endpoint of e_3 , and not using e_3 . Similarly, we define y as the first vertex of c_1 reached from the other endpoint of e_3 . Then x and y exist, and $x \neq y$ because e_2 lies in both c_1 and c_2 , then the (x, y) -segment of c_2 containing e_3 , together with the (y, x) -segment of c_1 containing e_1 , forms a cycle containing e_1 and e_3 .

□

Alternate characterization of 2-connectivity

Theorem 1.1

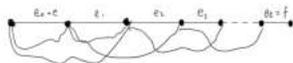
Let G be a graph with $|V(G)| \geq 3$. The following are equivalent:

1. G is 2-connected
2. G has no isolate vertices and every pair of edges lie in a common cycle.

3. Any two vertices lie in a common cycle.

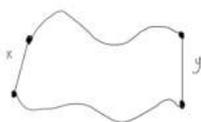
Proof

(a) \implies (b): Suppose G is 2-connected. Then G has no isolate vertices since $\delta(G) \geq 2$. Let e and f be any two edges in G . Since G is connected, there is a path P from an endpoint of e to an endpoint of f .

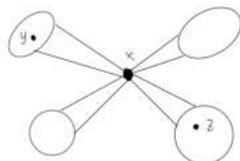


Then there exists a sequence of edges $e = e_1, e_2, \dots, e_k = f$ by the first lemma, for each i there is a cycle C_i containing e_i and e_{i+1} . Then applying the second lemma for each i in turn, we conclude there is a cycle containing $e = e_1$ and $f = e_k$.

(b) \implies (c): Assume G has no isolated vertices, and two edges lie in a common cycle. Let x and y be vertices in G . Since G has no isolated vertices, there exists edges e containing x and f containing y . Then there is a cycle containing e and f , and hence also x and y .



(c) \implies (a): Assume any two vertices lie in a common cycle. By assumption, $|V(G)| \geq 3$. Suppose on the contrary that x is a cut-vertex of G . Let y and z be vertices in distinct components of $G - x$. Then there can be no cycle containing y and z , since otherwise removing x can not disconnect y and z in C .



□

Q: What is a graph like if it is connected but not 2-connected?

1.2 Blocks

Definition 1.5. A **block** is a connected graph with no cut vertex.

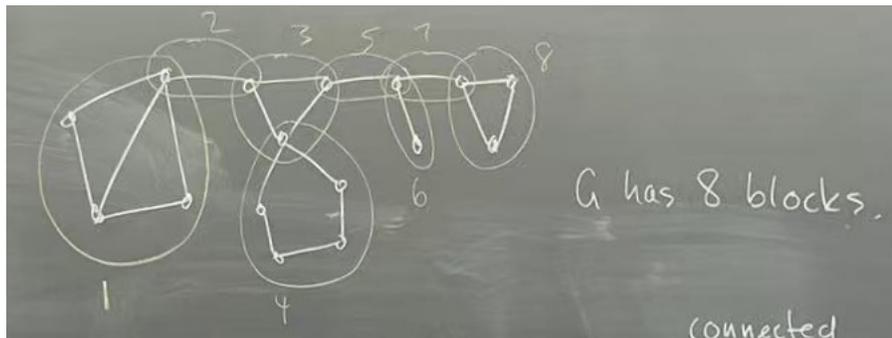
So a block is one of the following:

1. a single vertex (trivial block)
2. the graph K_2 (two vertices with an edge)
3. a 2-connected graph

Definition 1.6. A block of a connected graph G is a subgraph H of G and is maximal with respect to being a block.

- H is a block and,
- H is not contained in any large subgraph J of G that is also a block.

Example

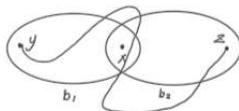


Lemma 1.5

Suppose blocks b_1 and b_2 in a connected graph G are distinct, and there exists a vertex $x \in v(b_1) \cap v(b_2)$, then x is a cut vertex of G .

Proof

Suppose on the contrary that $G - x$ is connected. There exist $y \in v(b_1) \setminus v(b_2)$ and $z \in v(b_2) \setminus v(b_1)$ by maximality of the distinct blocks b_1 and b_2 . Since $G - x$ is connected there exists a path P joining y and z in $G - x$.



□

Claim: $b_1 \cup b_2 \cup P$ is 2-connected.

Proof

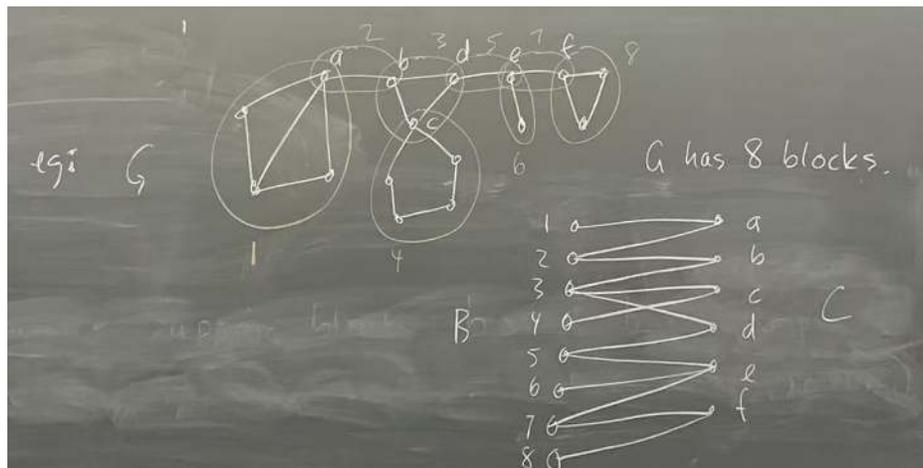
Suppose $w \in v(b_1 \cup b_2 \cup P)$. Then $b_1 - w$ is connected and $b_2 - w$ is connected, since b_1 and b_2 are blocks. Then $b_1 \cup b_2 \cup P - w$ is connected via x (if $x \neq w$) or via P (if $x = w$). So claim holds. But this contradicts the fact that b_1 is a block. (Also b_2 is a block.)

Hence, x is a cut vertex of G .

□

Definition 1.7. Let G be a connected graph. The **block-cutvertex forest** (BCF) of G is the bipartite graph with vertex classes B and C , where B is the set of blocks of G , and C is the set of cut vertices of G . The edge set is $\{bc : b \in B, c \in C, \text{ and } c \in b \text{ in } G\}$

Example



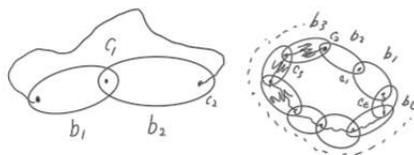
Theorem 1.2

Let G be a connected graph. The BCF of G is a tree.

Proof

The BCF of G is connected (Exercise!)

Suppose on the contrary that the BCF of G contains cycles. Let $c_1 b_1 c_2 b_2 \cdots c_t b_t$ be the shortest cycle in BCF of G then $t \geq 2$.



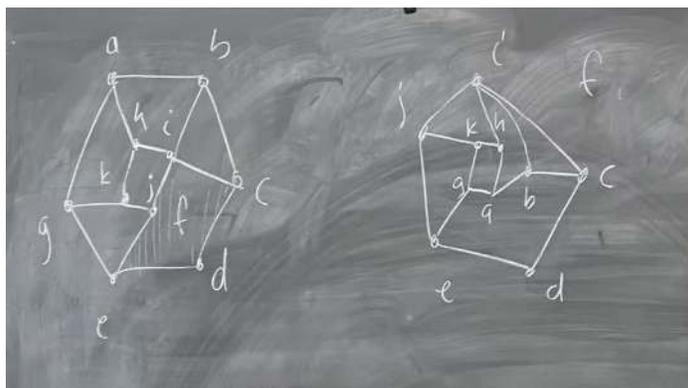
Then all $b_i - \{c_{i-1}, c_i\}$ are disjoint, since we chose a shortest cycle in BCF . There is a path P_i inside b_i from c_{i-1} to c_i for each i and these are all disjoint for the c_i for each i . Hence, $P = P_3 \cup P_4 \cup \cdots \cup P_t$ is a path from c_2 to c_t that does not contain c_1 . Hence, as in the previous proof $b_1 \cup b_2 \cup P$ is 2-connected, contradicting that b_1 is a block of G . Hence, BCF is a tree.

□

Definition 1.8. An end-block of a connected graph G is a block of G that has degree 1 in the BCF of G (i.e. a leaf of BCF). Note that every leaf of the BCF is a block, since any cutvertex separates the graph so is in at least 2 blocks. Hence, every graph with at least 2 blocks has one end-block.

Recall

That for any planar graph G , if \tilde{G} is a planar drawing of G and f is a face of \tilde{G} , then there exists a planar drawing \tilde{G}' of G in which f is the outer face.



1.2.1 K1 Theorem Proof

Theorem 1.3 K1

Let G be a connected graph, and suppose every block of G is planar. Then G is planar.

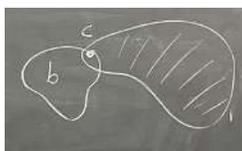
Proof

We use induction on the number t of blocks of G .

Base case: $t = 1$. Then G is itself a block. Hence, G is planar by assumption.

IH: Assume $t \geq 2$ and every connected graph with $\leq t - 1$ blocks, all of which are planar, is planar.

Let G with t blocks be given. Since $t \geq 2$, G has an end block b . Let c be the unique cutvertex of G in b .



By *IH*, the graph $H = (G - b) \cup \{c\}$ is planar, since it has $t - 1$ blocks, all of which were blocks of G and hence planar by assumption.

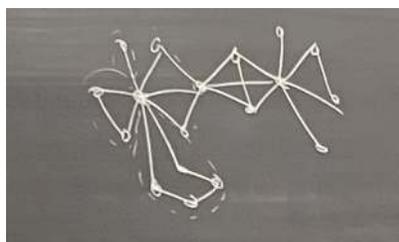
Also, b is planar since it is a block of G .

We take a planar drawing of H with c on the outer face (possible by lemma) and a planar drawing of b with c on the outer face (also possible by lemma) and glue the two drawings at c to get a planar drawing of G .

□

Using Blocks

We saw that if G is connected but not 2-connected then its blocks form a “tree-like” structure.



We can use the to prove general statements S about such graphs by

- Proving S for 2-connected graphs (and lines and single vertices)
- extend the proof of S to G by using induction on the numbers of blocks of G .

Example

Our Theorem K1

Recall

Contraction and Deletion of edges

Let G be a graph and let $e = xy$ be an edge of G .

The graph $G \setminus e$ is obtained by deleting the edge e from G has $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$.

The graph G / e obtained by contracting e in G has $V(G / e) = (V(G) \setminus \{x, y\}) \cup \{z\}$ (z is the image of e under contraction) and edge set

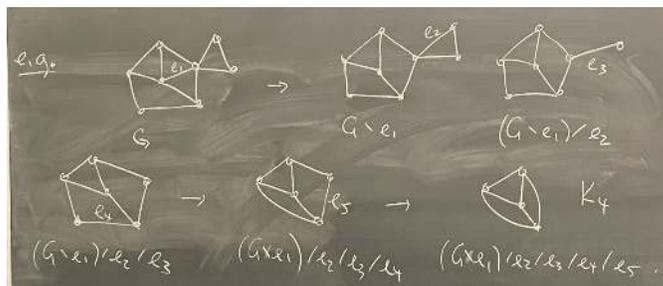
$$E(G / e) = \{uv \in E(G) : \{u, v\} \cap \{x, y\} = \emptyset\} \cup \{zu : ux \in E(G) \setminus \{e\} \text{ or } uy \in E(G) \setminus \{e\}\}$$

Note that $|E(G \setminus e)| = |E(G)| - 1$ and $|E(G / e)| \leq |E(G)| - 1$.

1.3 Minors

Definition 1.9. The graph H is said to be a minor of the graph G if H can be obtained from G by a sequence of edge deletions, edge contractions, and isolated vertices deletions.

Example

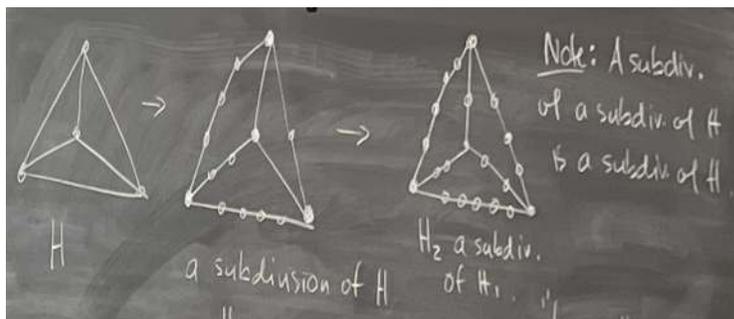


Hence, K_4 is a minor of G .

Definition 1.10. Let H be a graph. A subdivision J of H is any graph J obtained from H by replacing each edge of H by a path P_e of length at least 1 (whose endpoints are the endpoints of e)

such that

- $V(P_e) \cap V(P_f) = \{x\}$ whenever $e \cap f = \{x\}$
- $V(P_e) \cap V(P_f) = \emptyset$ whenever $e \cap f = \emptyset$

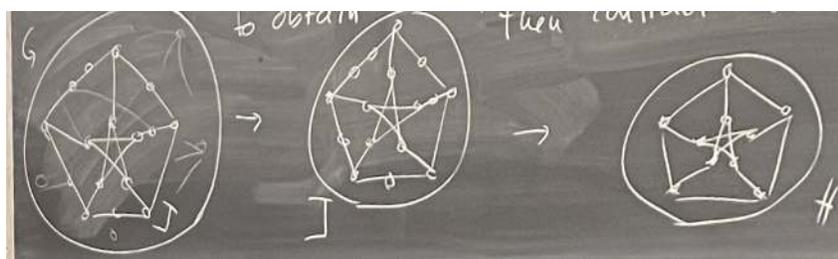


Note that $d_J(v) = d_H(v)$, if $v \in V(J) \cap V(H)$ “**branch**” vertices of J ,
 and $d_J(v) = 2$, if $v \in V(J) \setminus V(H)$ “**path**” vertices of J .

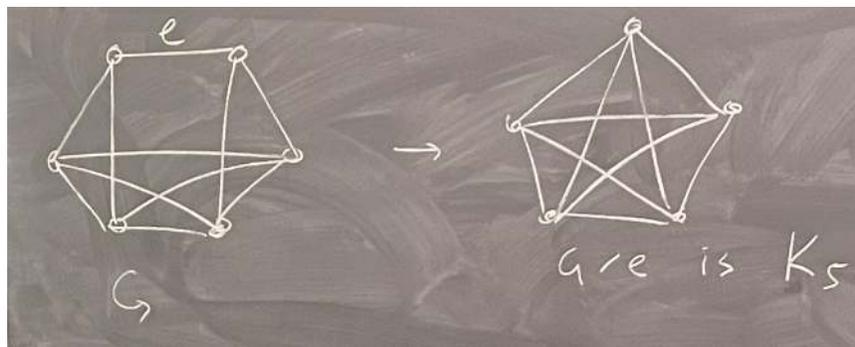
Compare:

- (a) G has H as a minor
- (b) G contains a subdivision of H .

(b) \implies (a): Suppose G contains a subdivision J of H . Start a sequence by deleting all edges of G not in J , to obtain J (plus isolated vertices — delete these). Then contract each P_e in J down to a single edge to set H .



(a) $\not\implies$ (b) in general: G contains K_5 as a minor, but G does not contain a subdivision J of K_5 , since J would need 5 vertices of degree 4 (the branch vertices) and G has only 4 vertices of degree ≥ 4 .



Definition 1.11. A **cubic** graph G is a graph in which every vertex has degree 3. (same as 3-regular)

Lemma 1.6

Let H be a cubic graph. If a graph G has H as a minor then G contains a subdivision of H .

Proof

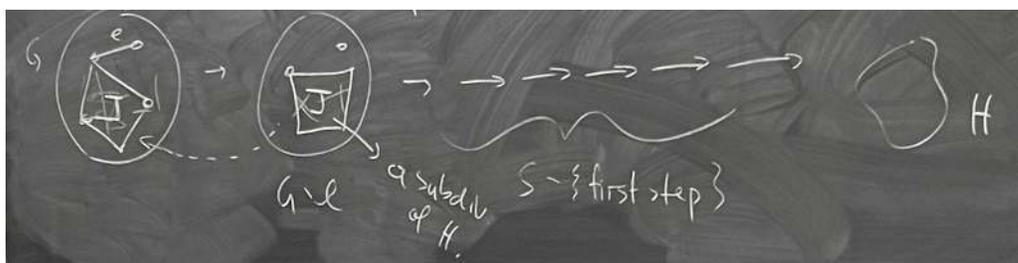
By induction on $|E(G)|$.

Base case: $|E(G)| = |E(H)|$. Then $G = H$. Then since H is a subdivision of itself, the statement is true.

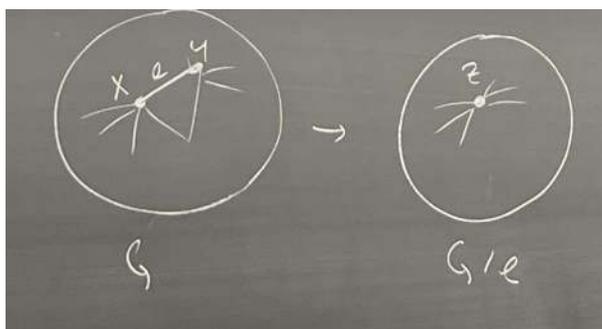
IH: Assume $|E(G)| > |E(H)|$ and for every graph G' with $|E(G')| \leq |E(G)| - 1$, that contains H as a minor, G' contains a subdivision of H .

Fix a sequence of edge deletions and contractions that take G to H .

Case 1: the first edge of S is a deletion of edge e , then $G \setminus e$ satisfies $|E(G \setminus e)| = |E(G)| - 1$. The rest of the sequence S except the first deletion step show that $G \setminus e$ has H as a minor. Hence, by IH we know that $G \setminus e$ contains a subdivision J of H . Hence, G contains the subgraph J as well.



Case 2: the sequence S begins with an edge contraction say taking G to G / e . Let $e = xy$



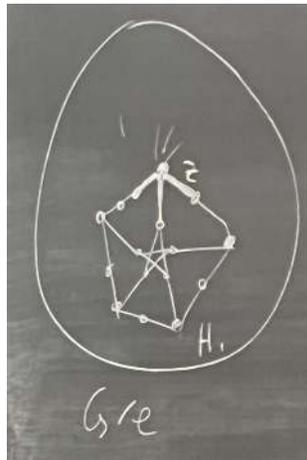
Then as before, say sequence S without its first step shows that G / e has H as a minor.

Since $|E(G / e)| < |E(G)|$, by IH we know that G / e contains a subdivision H_1 of H .

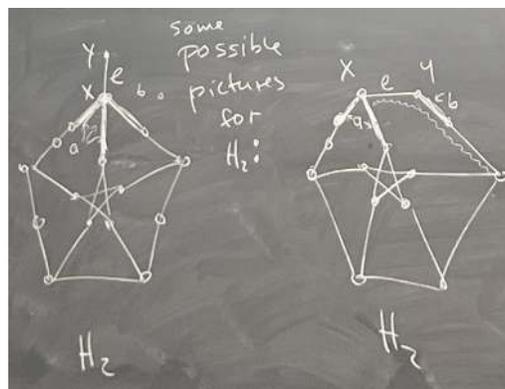
If the image z of e under contraction is not a vertex of H_1 , then H_1 is a subgraph of G as well, and we are done.

So we may assume that $z \in V(H_1)$. Hence, $d_{H_1}(z) = 3$.

Note that if $3 \geq a + b$ where a and b are non-negative integers, then one of a and b is ≤ 1 .



Let H_2 be the subgraph (minimal) of G such that $H_2 / e = H_1$ of the ≤ 3 edges of H_1 incident to z , one of x and y is incident to at most one of them in H_2 .



It follows that (in all cases)

H_2 contains a subdivision of H (possibly with one path length extended by 1).

Hence, G contains a subdivision of H .

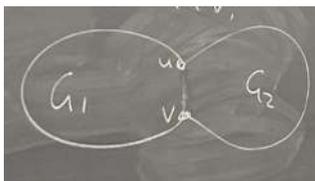
□

1.4 2-sum

Definition 1.12. Let G_1 and G_2 be graphs with at least 3 vertices.

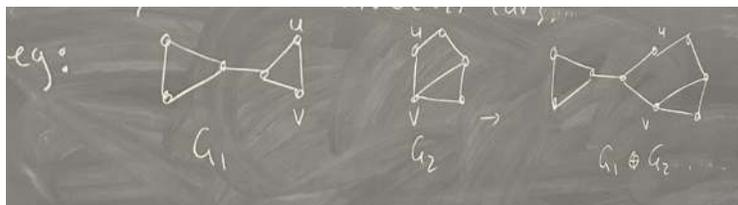
Suppose

- $V(G_1) \cap V(G_2) = \{u, v\}$ where $u \neq v$
- $uv \in E(G_1) \cap E(G_2)$



The **2-sum** $G_1 \oplus G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $(E(G_1) \cup E(G_2)) \setminus \{uv\}$.

Example



Note

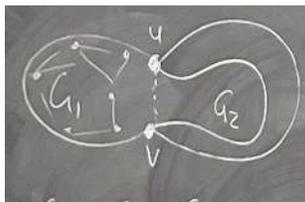
Any 2-sum of two cycles is a cycle: $C_k \oplus C_l = C_{k+l-2}$.

Note

In an 2-connected graph, every edge lies in a circle by characterization of 2-connected graphs: give e choose $f \neq e$ (exists since $\delta(G) \geq 2$) and take a cycle containing both.

Lemma 1.7

If $G = G_1 \oplus G_2$ via the edge uv , and if G_2 is 2-connected then G contains a subdivision of G_1 .

Proof

$$G = G_1 \oplus G_2$$

Since G_2 is 2-connected, there exists a cycle C in G_2 containing uv , then $C \setminus \{uv\}$ is a path in G from u to v that is disjoint from $V(G_1) \setminus \{u, v\}$. Hence $G \setminus \{uv\}$ contains a subdivision of G_1 .

□

Lemma 1.8

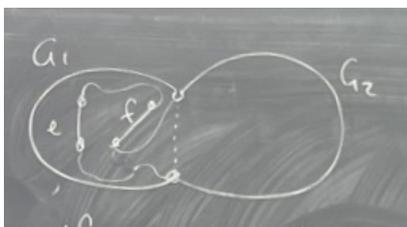
If $G = G_1 \oplus G_2$ (via uv) and both G_1 and G_2 are 2-connected then G is 2-connected.

Proof

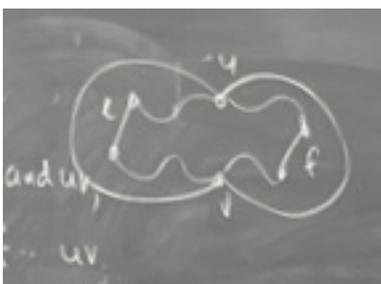
G has no isolated vertices since $\delta(G_1) \geq 2$ and $\delta(G_2) \geq 2$, and at most one edge is removed to form G .

Every pair of edges e, f in G lie in a common cycle in G .

- If $e, f \in E(G_1)$: there exists a cycle C in G_1 containing e, f .
If $uv \notin E(C)$, then C will do. If $uv \in E(C)$, then let C' be a cycle in G_2 containing uv and take $C \oplus C'$. This is a cycle in G containing e, f .



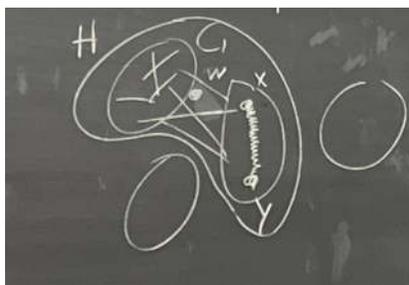
- If $e \in E(G_1)$ and $f \in E(G_2)$: take C_1 in G_1 containing e and uv , and C_2 in G_2 containing f and uv . Then $C_1 \oplus C_2$ is a cycle in G containing e, f .



□

Lemma 1.9

Let G be a 2-connected graph and suppose $\{x, y\}$ is a vertex cut of G . Let C_1 be a component of $G - \{x, y\}$. Then the graph H with vertex set $V(C_1) \cup \{x, y\}$ and edge set $\{wz \in E(G) \mid w, z \in V(H)\} \cup \{xy\}$ is 2-connected.



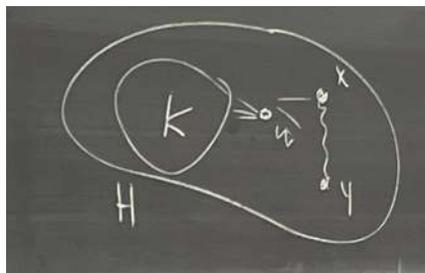
Proof

$|V(H)| \geq 3$ since $\{x, y\} \subseteq V(H)$ and C_1 is non empty. Also, H is connected since C_1 and G are connected.

Suppose w is a cutvertex of H , then we may assume $w \neq y$. Let k be a component of $H - w$ that does not contain y .

Then w is a cutvertex of G (that separates k from the rest of G).

This contradiction shows that H is 2-connected.



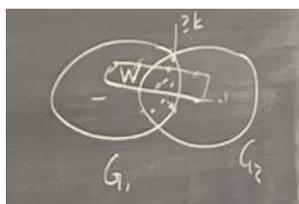
□

Lemma 1.10

Let G_1 and G_2 be k -connected graphs, where $|V(G_1) \cap V(G_2)| \geq k$. Then $G = G_1 \cup G_2$ is k -connected.

Proof

Clearly, $|V(G)| \geq |V(G_1)| \geq k + 1$. Let W be a set of $\leq k - 1$ vertices of G .



Then

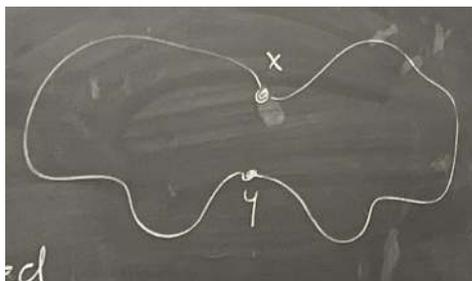
- $G_1 - W$ is connected since G_1 is k -connected.
- $G_2 - W$ is connected since G_2 is k -connected.
- $V(G_1) \cap V(G_2) - W$ is non-empty since $|W| < |V(G_1) \cap V(G_2)|$.

So, $G_1 \cup G_2 - W$ is connected. Hence, G has no vertex cut of size $\leq k - 1$.

□

Theorem 1.4 (Decomposition theorem)

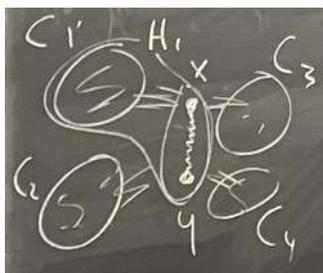
Suppose G is a 2-connected graph and $\{x, y\}$ is a vertex cut of G . Then there exists graphs G_1 and G_2 such that



1. $V(G_1) \cap V(G_2) = \{x, y\}$.
2. $xy \in E(G_1)$ and $xy \in E(G_2)$.
3. Both G_1 and G_2 are 2-connected.
4. If $xy \notin E(G)$, then $G = G_1 \oplus G_2$.
5. If $xy \in E(G)$, then $G - xy = G_1 \oplus G_2$.

Proof

Let C_1, C_2, \dots, C_r be the components of $G - \{x, y\}$. Then $r \geq 2$. For each i , Let H_i be the graph with vertex set $V(C_i) \cup \{x, y\}$ and edge set $\{wz \in E(G) \mid w, z \in V(H_i)\} \cup \{xy\}$.



By the previous lemma, each H_i is 2-connected, and $G_2 = H_2 \cup H_3 \cup \dots \cup H_r$ is 2-connected by our other lemma.

Set $G_1 = H_1$.

Then G_1 and G_2 have the given properties, and (4,5) hold by def of 2-sum.

□

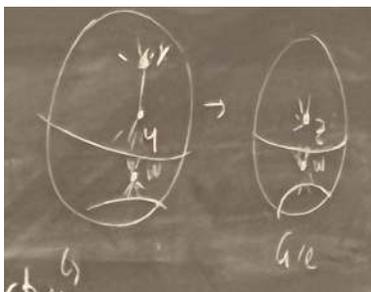
Corollary

Let G be a 2-connected graph with ≥ 4 vertices. Let $e \in E(G)$. Then either $G \setminus e$ is 2-connected or G / e is 2-connected (or both).

Proof

Fix $e = xy$. Suppose $G \setminus e$ is not 2-connected.

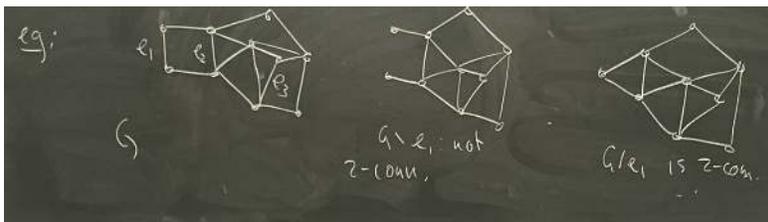
Let z be the image of e under contraction. Since $|V(G / e)| \geq 3$, and G / e is connected, G / e has a cutvertex, say w . If $w \neq z$, then w is a cutvertex of G , a contradiction. Hence, $w = z$, and so $\{x, y\}$ is a vertex cut of G .



Let G_1 and G_2 be as in the Decomposition theorem. Then G_1 and G_2 are 2-connected, and $G / e = G_1 \oplus G_2$. Hence, by an earlier result we know G / e is 2-connected.

□

Example



(Note: $G \setminus l_2$ is 2-connected, G / l_2 is not. Both $G \setminus l_3$ and G / l_3 are 2-connected.)

1.5 Kuratowski's Theorem

Reducing Kuratowski's Theorem to 3-connected graphs

1.5.1 K2 Theorem Proof

Theorem 1.5 K2

Supposes that every 3-connected graph that does not contain a subdivision of K_5 or $K_{3,3}$ is planar. Then every 2-connected graph that does not contain a subdivision of K_5 or $K_{3,3}$ is planar.

Proof

We use induction on the number of vertices of G .

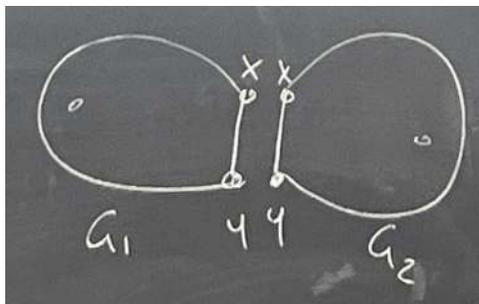
Base case: $|V(G)| \leq 5$. All graphs on ≤ 5 vertices except K_5 are planar. Hence, any G satisfying the assumptions is planar.

IH: Assume $|V(G)| \geq 6$ and every 2-connected graph H with $|V(H)| < |V(G)|$ that does not contain a subdivision of K_5 or $K_{3,3}$ is planar. Consider G is a 2-connected graph with no subdivision of K_5 or $K_{3,3}$.

- If G is 3-connected, then G is planar by the assumption of the theorem.
- Hence, we may assume there exists a vertex cut $\{x, y\}$ of G .

- By decomposition theory, there exists 2-connected graphs G_1 and G_2 each containing the edge xy , such that $G = G_1 \oplus G_2$ or $G - l = G_1 \oplus G_2$.

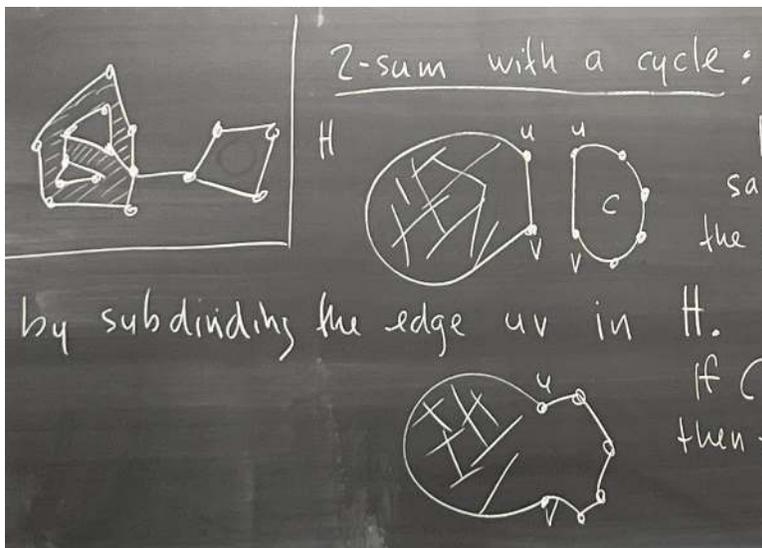
Then $|V(G_1)| < |V(G)|$ since $|V(G_2)| \geq 3$ (it is 2-connected). In $G_1 \oplus G_2$, there exists a subdivision of G_1 by an earlier lemma. Hence, G_1 does not contain a subdivision of K_5 or $K_{3,3}$. Since otherwise so would G .



Hence, by induction hypothesis, G_1 is planar. Similarly, G_2 is planar. Take a planar drawing of G , with edge xy on the outer face and a planar drawing of G_2 with edge xy on the outer face. Gluing these two planar drawings along the edge xy gives a planar drawing of G (if $xy \in E(G)$), or a planar drawing of $G \neq e$. Hence, G is planar by induction.

□

2-sum with a cycle



$H \oplus C$ is the same graph as the graph we get by subdividing the edge uv in H . If C has length k then the number of new path vertices is $k - 2$.

Theorem 1.6 (“Building 2 connected graph”)

Let G be a 2-connected graph. Then (at least) one of the following holds:

- (a) G is a cycle
- (b) There exists $e \in E(G)$ such that $G \setminus e$ is 2-connected.
- (c) G is the 2-sum of a 2-connected graph H with a cycle C (or: G is the result of subdividing an edge of a 2-connected graph G at least once).

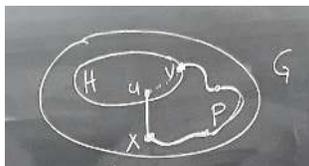
Proof

Call a graph “good” if it satisfies one of (a), (b), (c). Note that any good graph is 2-connected. (by an earlier lemma, for (c))

Let G be a 2-connected graph. Let H be a “good” subgraph of G with $|E(H)|$ as large as possible. Note H exists since G 2-connected $\implies G$ contains a cycle. If $|E(H)| = |E(G)|$ then $H = G$ and we are done. So suppose $|E(H)| < |E(G)|$. (★)

If $|V(H)| = |V(G)|$ then there exists $e \in E(G) \setminus E(H)$ and both endpoints of e are in $V(H)(= V(G))$. Then $G \setminus e$ contains H (a 2-connected graph) and $V(G \setminus e) = V(H)$. So $G \setminus e$ is 2-connected, hence G is good. Contradiction (★).

So we assume $|V(H)| < |V(G)|$. Since G is connected, there exist $u \in V(H)$ and $x \in V(G) \setminus V(H)$ where $ux \in E(G)$. Since u is not a cutvertex of G , there exists a path P in $G \setminus u$ from x to some $v \in V(H)$ with $V(P) \cap V(H) = \{v\}$.



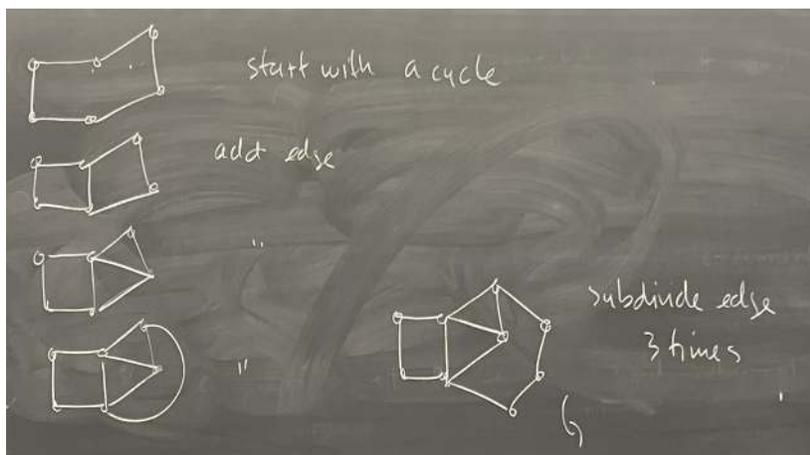
Let $H' = \begin{cases} H & \text{if } uv \in E(H) \\ H \cup \{uv\} & \text{if } uv \notin E(H) \end{cases}$ is 2-connected.

Let C be the cycle $P \cup \{u\} \cup \{xu, uv\}$. Then $J = H' \oplus C$ is a subgraph of G , is good (by (c)), and $|E(J)| = |E(H')| + (\text{length}(C) - 2) \geq |E(H)| + 1 \geq |E(H)|$. Contradiction (\star).

□

Example

Build G



Lemma 1.11

Let G be a 2-connected planar graph. In every planar drawing: every face is bounded by a cycle, and every edge is incident to exactly two distinct faces.

Proof

By induction on $|E(G)|$.

Since G has minimum degree ≥ 2 , $|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v) \geq |V(G)|$.

Base case: $|E(G)| = |V(G)|$, i.e. G is a cycle.

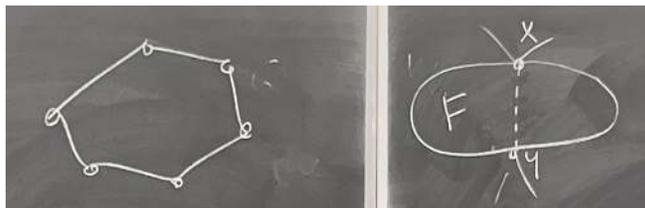
Statement is true for cycles

IH: $|E(G)| > |V(G)|$ and every 2-connected planar graph H with $|E(H)| < |E(G)|$ in every planar drawing of H , every face is bounded by a cycle, and every edge is incident to 2 faces.

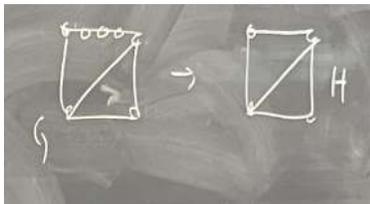
F_1 : a planar drawing of G

Case 1: There exists $e = xy \in E(G)$ such that $H = G \setminus e$ is 2-connected.

Then H has a planar drawing \tilde{H} , obtained by erasing e from the planar drawing of G . Then there is a face F of \tilde{H} that has both x and y on its boundary. By IH, every face of \tilde{H} is bounded by a cycle, and every edge is incident with 2 faces. Then replacing e into \tilde{H} , to get \tilde{G} partitions F (bounded by a cycle by IH) into 2 faces, each bounded by a cycle, and e is incident to both. So the drawing of G is as claimed.



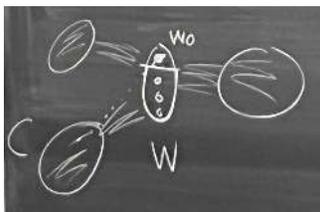
Case 2: G is obtained by subdividing an edge of a 2-connected graph H : then a planar drawing of H can be obtained from \tilde{G} via the subdivision. Then \tilde{H} is “nice” by IH, so \tilde{G} is also nice.



□

Lemma 1.12

Let G be a k -connected graph. Suppose G has a vertex cut W of size k . Then for every component C of $G - W$, and every $w \in W$, there is an edge in G from w to a vertex in C .

**Proof**

Suppose on the contrary that some $w_0 \in W$ has no edge to some component C , then $W \setminus \{w_0\}$ is a vertex cut of G that separates C from $G - (W \setminus \{w_0\})$, contradicting that G is k -connected.

□

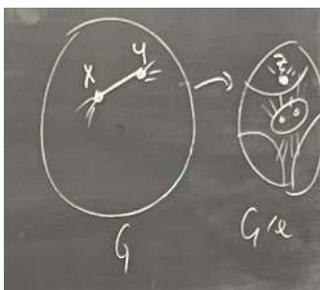
Definition 1.13. Let G be a 3-connected graph, an edge e of G is called **contractible** if G / e is also 3-connected.

Theorem 1.7

Every 3-connected graph with at least 5 vertices has a contractible edge.

Proof

Suppose on the contrary that G / e is not 3-connected for any edge e . Since $|V(G)| \geq 5$, every G / e will have at least 4 vertices, and will be connected. Hence, every G / e must have a vertex cut of size ≤ 2 .



Let $e = xy$ be fixed. Let z denote the image of e in G / e .

- Any vertex cut of G / e must contain z , otherwise it would be a vertex cut of G of size ≤ 2 , contradicting that G is 3-connected.
- Any vertex cut of G / e of size ≤ 2 has size exactly 2, otherwise $\{x, y\}$ is a vertex cut of G .

(\star) Hence, for every edge $e = xy$ in G , there exists $w \in V(G) \setminus \{x, y\}$ such that $\{z, w\}$ is a vertex cut of G / e , this implies that $\{x, y, w\}$ is a vertex cut of G ...

We choose

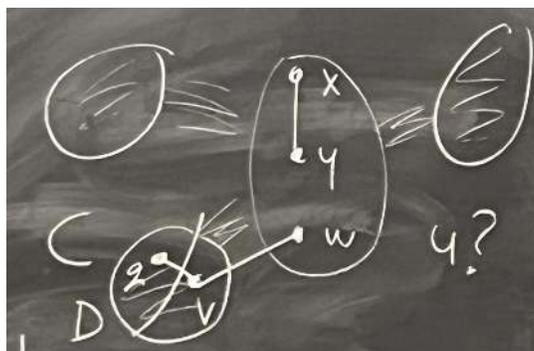
- the edge xy of G .
- the vertex $w \in V(G) \setminus \{x, y\}$
- the component C of $G - \{x, y, w\}$

such that $|V(C)|$ is as small as possible.

By the lemma, we know that w has a neighbor v in C . Hence, there exists (by (\star)) a vertex $u \in V(G) \setminus \{v, w\}$ such that $\{v, w, u\}$ is a vertex cut of G . We'll use edge wv and vertex u to show we made the wrong choice.

We claim that some component D of $G - \{v, w, u\}$ is disjoint from $\{x, y\}$. This is true because $G - \{x, y, w\}$ has at least 2 components and x and y are joined by an edge.

We'll show (wv, u, D) contradicts our choice of (xy, w, C) .



By our previous lemma, we know v has a neighbor q in D . Then $q \in V(C)$ since $q \notin \{x, y, w\}$ and C is a component of G , so all neighbors of v are in $V(C) \cup \{x, y, w\}$.

So $D \cap C \neq \emptyset$, hence $D \subseteq C$ since D is disjoint from $\{x, y, w\}$. Since $v \in V(C) \setminus V(D)$, we find $D \subset C$ and so $|V(D)| < |V(C)|$, contradicting our choice of C .

□

Lemma 1.13

Let e be an edge of a graph G and suppose G / e contains a subdivision of K_5 . Then G contains a subdivision of K_5 or a subdivision of $K_{3,3}$.

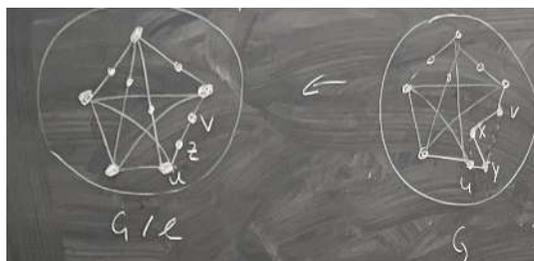
Proof

Let z be the image of the edge $e = xy$ under contraction. Let K be a subdivision of K_5 in G / e .

If $z \notin V(K)$, then K is also a subgraph of G , and we are done.



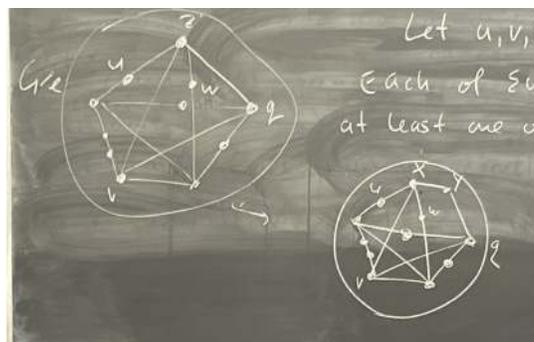
Case 1: z is a path vertex in K .



Let u and v be the neighbors of z in K . Then each of u and v was adjacent to at least one of $\{x, y\}$ in G .

Each of these possibilities gives a subdivision of K_5 in G .

Case 2: z is a branch vertex in K .



Let u, v, w, q be the neighbors of z in K . Each of $\{u, v, w, q\}$ was a neighbor of at least one of $\{x, y\}$ in G .

If at least 3 of $\{u, v, w, q\}$ were joined to say x , then we find a subdivision of K_5 in G (several possibilities here).

If two of $\{u, v, w, q\}$ were joined to x and two are joined to y ,



We find a subdivision of $K_{3,3}$ in G .

□

1.5.2 K3 Theorem Proof

Theorem 1.8 K3 (Kuratowski's Theorem for 3-connected graphs)

Let G be a 3-connected graph that does not contain a subdivision of K_5 or $K_{3,3}$. Then G is planar.

Proof

By induction on $|V(G)|$,

Base Case: $|V(G)| \leq 5$. Then since K_5 is the only non-planar graph with ≤ 5 vertices, G is planar.

IH: Assume $|V(G)| \geq 6$ and every 3-connected graph with fewer than $|V(G)|$ vertices that does not contain a subdivision of K_5 or $K_{3,3}$ is planar.

Let G be given with $|V(G)| \geq 6$, 3-connected no subdivision of K_5 or $K_{3,3}$.

Since G is 3-connected and has ≥ 5 vertices, it has a contractible edge e . Then $H = G / e$ is 3-connected and $|V(H)| < |V(G)|$.

Proposition

Claim: $H = G / e$ does not contain a subdivision of K_5 or $K_{3,3}$.

Proof

- If G / e contains a subdivision of K_5 , then by our earlier lemma, G contains a subdivision of K_5 or $K_{3,3}$.
- If G / e contains a subdivision of $K_{3,3}$, then in particular G / e has $K_{3,3}$ as a minor. Hence, G has $K_{3,3}$ as a minor. Hence, by an even earlier lemma since $K_{3,3}$ is cubic we found G contains a subdivision of $K_{3,3}$.

These contradictions complete the proof of the claim.

□

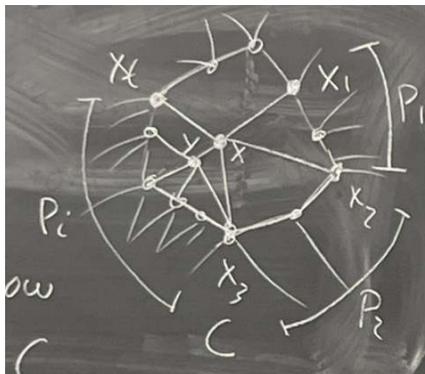
Hence, by IH we know $H = G / e$ is planar.

Let \tilde{H} be a planar drawing of $H = G / e$. Let $e = xy$ and let z denote the image of e under contraction.

Erase z (and its incident edges) from \tilde{H} to set a planar drawing of $H - z (= G - \{x, y\})$.

Since H is 3-connected, $H - Z$ is 2-connected. Then by an earlier lemma we know that $\tilde{H} - z$ every face is bounded by a cycle. So the face of $\tilde{H} - z$ that contained z in \tilde{H} is bounded by a cycle C .

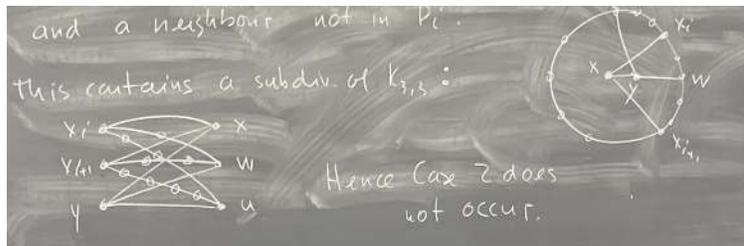
Since all neighbours of x and of y in G became neighbours of z in $H = G / e$, we know that all such neighbours are vertices of C . Let x_1, \dots, x_t denote the neighbours of x (except y) in order as they appear around C . Let P_i denote the $(x_i - x_{i+1})$ path on C , for $1 \leq i \leq t$ (where $t + 1 := 1$).



We consider where on C the neighbours of y lie.

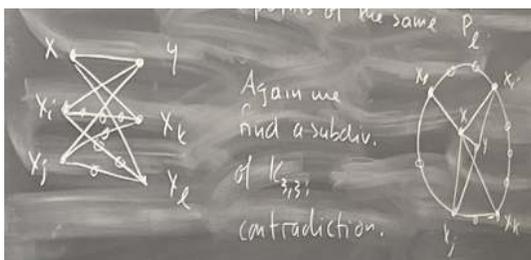
Case 1: All neighbours $N(y) \setminus \{x\}$ lie in the same P_i . Then we can complete two drawing to a planar drawing of G .

Case 2: y has a neighbour in the interior of same P_i and a neighbour not in P_i . This contains a subdivision of $K_{3,3}$.



Hence, Case 2 does not occur.

Case 3: y has neighbours x_i and x_j where x_i and x_j are not both endpoints of the same P_i .



Case 4: y has 3 neighbours $x_i, x_j, x_k (= x_1, x_2, x_3)$ Then we find a subdivision of K_5 .



Hence, Case 1 is the only possibility, implies G is planar.

□

1.5.3 Kuratowski's Theorem Proof

Theorem 1.9 Kuratowski's Theorem for general graphs

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proof

(\Rightarrow)

From Math 239, K_5 and $K_{3,3}$ are both non-planar, so any graph with a subdivision of K_5 or $K_{3,3}$ as a subgraph is non-planar.

(\Leftarrow)

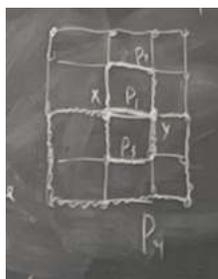
Suppose G does not contain a subdivision of K_5 or $K_{3,3}$.

- If G is 3-connected, it is planar by Theorem K3.
- If G is 2-connected, it is planar by Theorem K2.
- If G is connected but not 2-connected, since every block of G is a dot or line or 2-connected, and the 2-connected blocks are planar by K2, all blocks of G are planar. Hence, by theorem K1, G is planar.
- If G is not connected, consider each component separately.

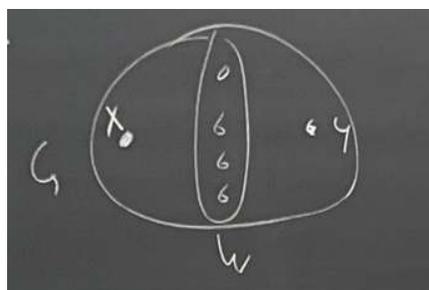
□

1.6 Connectivity and Menger's Theorem

Definition 1.14. A set of paths $\{P_1, P_2, \dots, P_k\}$ in a graph G , all with the same endpoints x and y is said to be internally disjoint if $V(P_i) \cap V(P_j) = \{x, y\}$ for each $i \neq j$.



Definition 1.15. Let x and y be vertices in a graph G . We say a vertex cut W of G is said to separate x and y in G if x and y are in different components of $G - W$.



(Note then that $x, y \notin W$ and $xy \notin E(G)$)

Recall

The notion of sets of internal disjoint paths joining x and y (we'll write “ID (x, y) -paths”)



Note that if there exists a set of k ID (x, y) -paths in G , then there is no vertex cut W with $|W| \leq k - 1$ that separates x and y in G .

Theorem 1.10 (Menger’s theorem)

Let a and b be non-adjacent vertices in a graph G . Let s be the minimum size of a vertex cut that separates a and b in G . Then G contains a set of s ID (a, b) -paths.

Proof

The statement is clearly true, if $s \leq 1$, so assume $s \geq 2$.

We apply induction on $|E(G)|$.

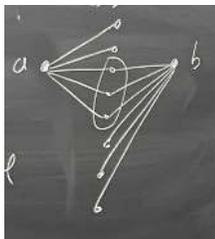
Base Case: $|E(G)| = 0$. Then $s = 0$ and the statement is true.

Induction Step: Assume $|E(G)| \geq 1$, and for every graph H with fewer $|E(H)| < |E(G)|$, and every pair of non-adjacent vertices x, y in H , there exists t ID (x, y) -paths, where t is the min size of a vx cut separating x and y in H .

Given G , and vertices a and b with $ab \notin E(G)$.

Case 1: Every edge of G is incident to a or b .

Let $N(a) = \{x \in V(G) : ax \in E(G)\}$ and $N(b) = \{y \in V(G) : by \in E(G)\}$.



Then, $s = |N(a) \cap N(b)|$. So $\{a \cup b : u \in N(a) \cap N(b)\}$ is a set of s ID (a, b) -paths in G as required.

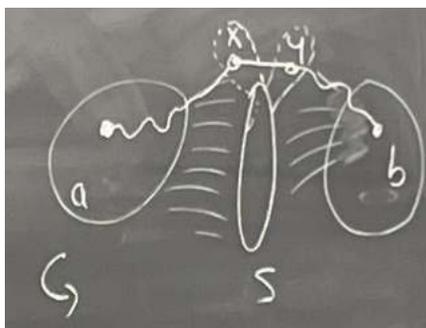
Case 2: There exists an edge $xy = l$ in G with $\{x, y\} \cap \{a, b\} = \emptyset$ Let $H = G \setminus e$.

Let S be a vertex cut of H of minimum size that separates a and b in H .

- If $|S| = s$, then by IH H contains a set of s ID (a, b) -paths, and we are done.
- So we may assume $|S| < s$

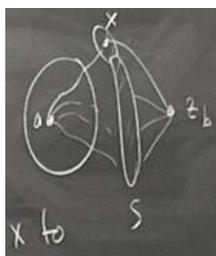
Note that $S \cup \{x\}$ is a vertex cut separating a and b in G .

Then $|S \cup \{x\}| \geq s$, by definition of s , hence $|S| = s - 1$.



We know there exists a (a, b) -path in $G - S$ since $|S| < s$, so it must contain the edge xy . Assume WLOG x is closer than y to a on this path.

Let G_b be the graph obtained from, G by contracting, one by one all edges in the component C_b of $G - (S \cup \{x\})$ that containing b . Let z_b be the image under contraction then $|E(G_b)| < |E(G)|$, since (in particular the edges of P from x to b disappear).



If T is a vertex cut separating a and z_b in G_b , then T is a vertex cut separating a and b in G .

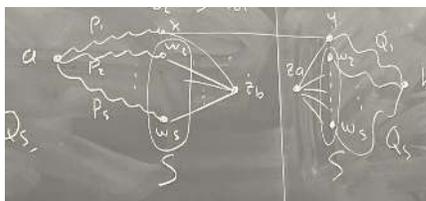
Hence, $|T| \geq s$ by definition of s .

i.e. the min size of a vertex cut separating a and z_b in G_b is at least s .

Hence, by IH there exists a set of $\geq s$ ID (a, z_b) -paths in G_b .

Since $N(z_b)_{G_b} = S \cup \{x\}$, there are exactly s of these paths $P_1 \dots P_s$, WLOG P_1 contains x and P_i contains $w_i \in S$ for $i = 2 \dots s$.

Similarly, we can find s ID (z_a, b) -paths in the graph G_a , call them $Q_1 \dots Q_s$.



Claim: $P_1 \cup Q_1 \cup \{xy\}, P_2 \cup Q_2, \dots, P_s \cup Q_s$ form a set of s ID (a, b) -paths in G . Since each $V(P_i) \setminus (S \cup \{x, z_b\})$ is contained in the component C_a of a in $H - S$, and similarly for C_b and Q_i , we know there are ID.

□

1.6.1 Menger-Whiteley Theorem and Applications

Theorem 1.11 (Menger-Whiteley Theorem)

Let G be a graph with at least 2 vertices. Then G is k -connected if and only if for every pair of vertices a and b there exists a set of k ID (a, b) -paths.

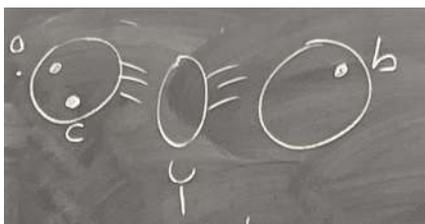
Proof

True for $k = 1$, so we may assume $k \geq 2$.

(\Rightarrow) Assume G is k -connected, let $a, b \in V(G)$.

- If $ab \notin E(G)$ by Menger's theorem there exists a set of k ID (a, b) -paths.
- If $e = ab \in E(G)$, then (claim) $G \setminus e$ is $(k - 1)$ -connected.

Suppose not then there exists a vertex cut Y of $G \setminus e$ with $|Y| \leq k - 2$. Then a and b are in different components of $G' - Y$, otherwise Y is a vertex cut of G separating a and b . WLOG same component of $G' - Y$ not containing b has vertex $c \neq a$.

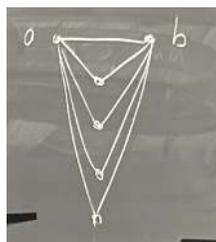


Then $Y \cup \{a\}$ is a vertex cut of G separating b and c , contradicting that G is k -connected.

By Menger's Theorem, G' contains a set of $k - 1$ ID (a, b) -paths. These together with $e = ab$ satisfy the conclusion.



(\Leftarrow) Assume G is such that every pair of vertices is joined by a set of k ID paths. Then G has no vertex cut of size $\leq k - 1$. To see that $|V(G)| \geq k + 1$, let $a, b \in V(G)$ of the k ID (a, b) -paths, at least $k - 1$ of them have an internal vertex, all of which are distinct. Hence, $|V(G)| \geq 2 + k - 1 = k + 1$.

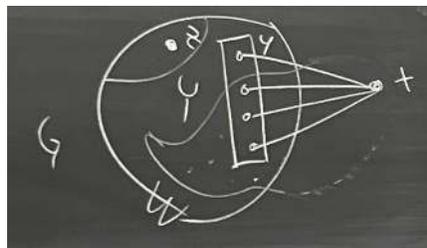


□

1.6.2 Some Useful Lemmas

Lemma 1.14 (Extension Lemma)

Let G be a k -connected graph. Let $Y \subset V(G)$ be a set of vertices with $|Y| = k$. Then the graph H with $V(H) = V(G) \cup \{x\}$ and $E(H) = E(G) \cup \{xy : y \in Y\}$ is also k -connected. (where x is a new vertex)



Proof

$$|V(H)| > |V(G)| \geq k + 1,$$

Suppose W is a vertex cut of H , and suppose on the contrary $|W| \leq k - 1$. If $x \in W$ then $W \setminus \{x\}$ is a vertex cut of G of size $\leq k - 2$, contradicting k -connectivity of G . So $x \notin W$. Let $z \in V(G)$ be a vertex in a component of $H - W$ that does not contain x .

Let y be a vertex of $Y \setminus W$ (which exists because $|Y| = k > |W|$). Then y is in the component of x in $G - W$, hence W is a vertex cut of G separating y and z . Again this contradicts k -connectivity of G .

□

Definition 1.16. Let G be a graph, $x \in V(G)$, and $Y \subseteq V(G) \setminus \{x\}$. A **(x, Y) -fan** in G is a set $S = \{P_1, P_2, \dots, P_k\}$ of k paths in G from x to Y , such that $k = |Y|$ and $V(P_i) \cap V(P_j) = \{x\}$ for each $i \neq j$.

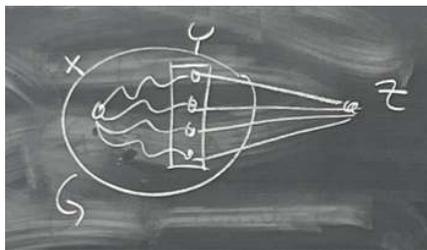


Lemma 1.15 (Fan Lemma)

Let G be a k -connected graph, let $x \in V(G)$, and let $Y \subseteq V(G) \setminus \{x\}$ where $|Y| = k$. Then G contains a (x, Y) -fan.

Proof

Let H be the graph with $V(H) = V(G) \cup \{z\}$ and $E(H) = E(G) \cup \{zy : y \in Y\}$. By our extension lemma, H is k -connected. By the Menger-Whitney Theorem, there exists a set of k internally disjoint (x, z) -paths $\{P_1, P_2, \dots, P_k\}$ in H .



Since $N(z) = Y$, each path P_i must end with an edge yz where $y \in Y$, and these are all distinct. Hence, $\{Q_1, Q_2, \dots, Q_k\}$ is a (x, Y) -fan where $Q_i = P_i - z$.

□

Note

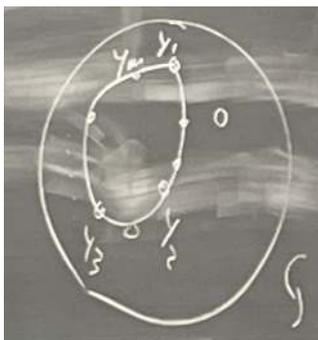
The fan lemma is if and only if. If G contains a (x, Y) -fan for every $x \in V(G)$ and every $Y \subseteq V(G) \setminus \{x\}$ with $|Y| = k$, then G is k -connected.

Lemma 1.16 (Cycle Lemma)

Let G be a k -connected graph, $k \geq 2$. Let $Y \subset V(G)$ be such that $|Y| = k$. Then G contains a cycle C with $Y \subseteq V(C)$.

Proof

Let C_0 be a cycle contains as many vertices of Y as possible, say $V(C_0) \supseteq \{y_1, \dots, y_m\}$. Then $m \geq 2$ by our characterization of 2-connectivity. If $m = k$ we are done, so suppose $m < k$.



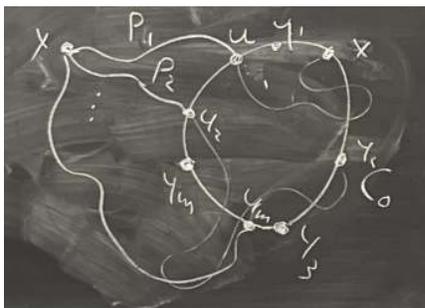
Case 1. $V(C_0) = \{y_1, \dots, y_m\}$

Let $y \in Y \setminus V(C_0)$. By the fan lemma for m -connected graph. (We know G is k -connected, hence m -connected) we can find a $(y, \{y_1, \dots, y_m\})$ -fan in G call it $\{P_1, P_2, \dots, P_m\}$. Then $C = C_0 \setminus (y_1, y_2) \cup P_1 \cup P_2$ is a cycle in G containing $\geq m + 1$ vertices of Y , contradicting our choice of C_0 .



Case 2. $V(C_0) \setminus \{y_1, \dots, y_m\} \neq \emptyset$. Choose $x \in V(C_0) \setminus \{y_1, \dots, y_m\}$.

Let $\{P_1, \dots, P_m\}$ be a $(y, \{y_1, \dots, y_m\})$ -fan in G . (Note: $m + 1 \leq k$)



For $0 \leq i \leq m$, let u_i be the first vertex of P_i on C_0 when going from y to C_0 . (Note: $u_i = y_i$ is possible).

So we set $m + 1$ u_i 's and m y_i 's on $V(C_0)$.



By pigeonhole principle, some segment u_i to u_{i+1} on C_0 contains two vertices u_j and u_l . Then replacing the (u_j, u_l) -segment inside the (u_i, u_{i+1}) -segment of C_0 with the (u_j, y) -segment of P_j and the (y, u_l) -segment of P_l gives a cycle containing y_1, \dots, y_m and y . Again contradicting our choice of C_0 .

□

Using Fan Lemma and Cycle Lemma

Theorem 1.12

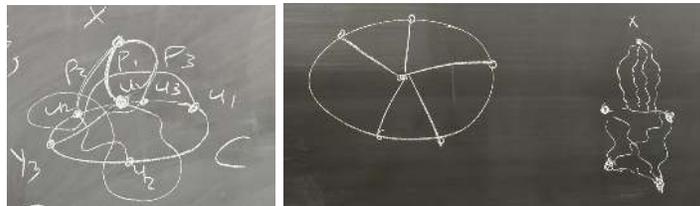
Every 3-connected graph contains a subdivision of K_4 .



Proof

Fix $x \in V(G)$. Then $G - x$ is 2-connected. Then by the Cycle Lemma, $G - x$ contains a cycle C .

Let $y_1, y_2, y_3 \in V(C)$ be three distinct vertices. By the Fan Lemma, there exist an (x, y) -fan where $y = \{y_1, y_2, y_3\}$ say P_1, P_2, P_3 . Let Q_i be the segment of P_i from x to the first vertex u_i on $C \cap P_i$ for each i . Then $C \cup Q_1 \cup Q_2 \cup Q_3$ is a subdivision of K_4 .



□

Matching

Definition 2.1. A **matching** in a graph G is a set M of disjoint edges in G .

A **maximum matching** is a matching of maximum size in G . We denote the size of a maximum matching in G is written $\nu(G)$.

The matching M **saturates** the vertex $v \in V(G)$ if v is incident to an edge of M , otherwise we say v is **exposed**.

We say M is a **perfect** matching if it satisfies $\nu(G)$.

A **vertex cover** W of a graph G is a set $W \subseteq V(G)$ such that $G - W$ has no edges.

2.1 Greedy Algorithm for Matching

Algorithm 1: Greedy Matching Algorithm

Input: Graph $G = (V, E)$

Output: Matching $M \subseteq E$

$M \leftarrow \emptyset, H \leftarrow G;$

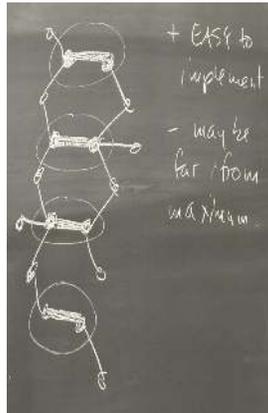
while $E(H) \neq \emptyset$ **do**

 choose any edge $xy \in E(H);$

$M \leftarrow M \cup \{xy\};$

$H \leftarrow H - \{x, y\};$

return $M;$

Example**Lemma 2.1**

The greedy algorithm always finds a matching in G of size $\geq \frac{\nu(G)}{2}$.

Proof

Let H be a matching obtained by Greedy. Then the set $V(H)$ of vertices saturated by H is a vertex cover of G .

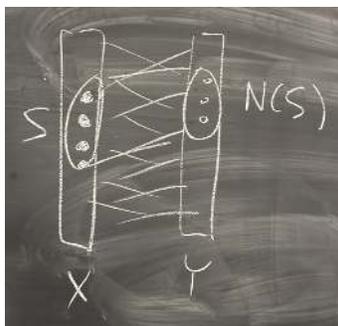
Let M^* be a maximum matching in G . Since each edge of M^* has a vertex in $V(H)$, and no two edges of M^* share the same vertex, we find $|M^*| \leq |V(H)| = 2|M|$. Thus, $|M| \geq \frac{|M^*|}{2} = \frac{\nu(G)}{2}$.

□

2.2 Hall's Theorem**Bipartite Matching**

Let G be a bipartite graph with vertex classes X and Y .

Note that if $S \subseteq X$ is such that $|N(S)| < |S|$, then there is no matching saturating X .

**Theorem 2.1 (Hall's Theorem)**

Let G be a bipartite graph with vertex classes X and Y . Then G has a matching saturating X if and only if

$$(\star) \quad |N(S)| \geq |S| \text{ for every } S \subseteq X. \text{ (Hall's Condition)}$$

Proof

(\Rightarrow): see note

(\Leftarrow): Assume G has (\star). We use induction on $|X|$.

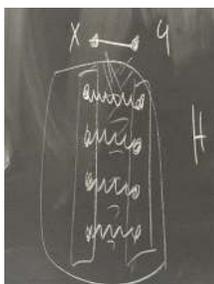
Base Case: If $|X| = 1$, then G has a matching of size 1.

IH: Assume $|X| \geq 2$, and for every bipartite graph H with vertex classes X' and Y' , where $|X'| < |X|$, that satisfies (\star) for H, X' , there exists a matching saturating X' .

Given G

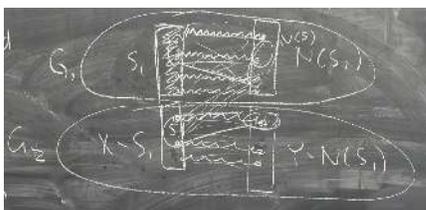
Case: 1 Every $S \subset X$ with $\emptyset \neq S \neq X$ satisfies $|N(S)| > |S|$.

Let $xy \in E(G)$, let $H = G - \{x, y\}$. For every $S \subseteq X' = X - \{x\}$, we have $N_H(S) = N_G(S) \setminus \{y\}$. So, $|N_H(S)| \geq |N_G(S)| - 1 \geq |S| - 1 = |S|$. By the case 1 condition for S . So $|N_H(S)| \geq |S|$. This says Hall's condition holds in H . So by IH, we get H has a matching M' saturating X' . Then $M = M' \cup \{xy\}$ is a matching in G saturating X .



Case: 2 There exists S_1 with $\emptyset \neq S_1 \neq X$ such that $|N(S_1)| = |S_1|$.

Let G_1 be the subgraph of G induced by $S_1 \cup N(S_1)$. Then (\star) for each $S \subseteq S_1$ we have $|N_{G_1}(S)| \geq |S|$ by (\star) for G , and because $N_{G_1}(S) = N_G(S)$ because of our definition of G_1 . So by IH (since $|S_1| < |X|$) G_1 has a matching M_1 saturating S_1 .



Let G_2 be the subgraph of G induced by $(X - S_1) \cup (Y - N(S_1))$. Take an arbitrary $S \subseteq X \setminus S_1$. Then $N_{G_2}(S) = N_G(S \cup S_1) \setminus N(S_1)$.

So, $|N_{G_2}(S)| = |N_G(S \cup S_1)| - |N_G(S_1)|$ But $|N_G(S \cup S_1)| \geq |S \cup S_1|$ by (\star) for G , and $|N_G(S_1)| = |S_1|$ by our choice of S_1 . So,

$$|N_{G_2}(S)| \geq |S \cup S_1| - |S_1| = |S|.$$

So, (\star) holds for G_2 . By IH for G_2 we get a matching M_2 in G_2 saturating $X - S_1$.

Then $M_1 \cup M_2$ is a matching in G saturating X .

□

Theorem 2.2

Any regular bipartite graph G of positive degree k has a perfect matching.

Proof

Let X and Y be the vertex classes of G .

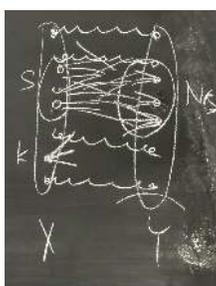
Let $S \subseteq X$. Let $E(S, N(S))$ be the set of edges of G joining S to its neighborhood $N(S)$.

Then $|E(S, N(S))| = k|S|$ since G is k -regular.

Since G is k -regular, $|E(S, N(S))| = k|N(S)|$, and $|E(S, N(S))| \leq |E(X, N(S))| = k|N(S)|$.

So,

$$k|S| = |E(S, N(S))| \leq k|N(S)|.$$



Therefore, since $k > 0$, $|N(S)| \geq |S|$. By Hall's Theorem, G has a matching saturating X .

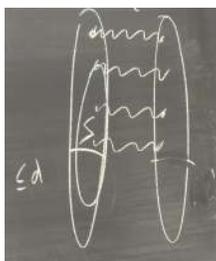
Since $E(G) = k|X| = k|Y| \implies |X| = |Y|$, we see a perfect matching exists.

□

Theorem 2.3 (“default version of Hall’s Theorem”)

Let $d \geq 0$ and let G be a bipartite graph with vertex classes X and Y , Then G has a matching of size $\geq |X| - d$ if and only if

$$(\star\star) |N(S)| \geq |S| - d, \forall S \subseteq X.$$



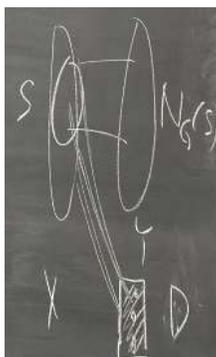
Proof

(\Rightarrow)

If G has a matching M of size $\geq |X| - d$, then for every $S \subseteq X$, M matches at least $|S| - d$ vertices of S to vertices in $N(S)$.

(\Leftarrow)

Assume G has $(\star\star)$. Form the graph H by adding a set D to Y and joining all vertices in D to all vertices in X , where $|D| = d$.



Let $\emptyset \neq S \subseteq X$. Then

$$N_H(S) = N_G(S) \cup D \implies |N_H(S)| = |N_G(S)| + d \geq |S| - d + d = |S| \quad (\star\star)$$

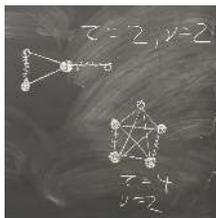
Hence, by Hall's Theorem, H has a matching saturating X . At most d vertices of M have a vertex in D , so the rest of M forms a matching of size $|X| - d$ in G .

□

Definition 2.2. We write $\tau(G)$ for the minimum size of a vertex cover of G .

Note

In any graph $\tau(G) \geq \nu(G)$.



Theorem 2.4 (König's Theorem)

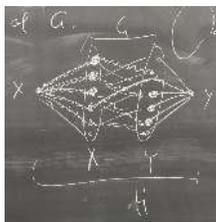
In a bipartite graph G , $\tau(G) = \nu(G)$.

Proof

Let X and Y be the vertex classes of G .

Let x, y be new vertices, and form H

$$V(H) = V(G) \cup \{x, y\}, \text{ and } E(H) = E(G) \cup \{xv : v \in X\} \cup \{yw : w \in Y\}.$$



Let W be a vertex cut separating x and y in H . Then $W \subseteq V(G)$ and W is a vertex cover of G , so

$$|W| \geq \tau(G).$$

Then by Menger's Theorem there is a set of $\geq \tau(G)$ ID (x,y) -paths in H . So taking the second edge on each of these paths gives a matching in G of size $\geq \tau(G)$. Hence, $\nu(G) \geq \tau(G)$.

Therefore, since $\tau(G) \geq \nu(G)$, we have $\tau(G) = \nu(G)$.

□

Definition 2.3. A set $U \subseteq V(G)$ in G is called **independent** in G if there is no edge in G joining two vertices of U . The maximum size of an independent set in G is denoted $\alpha(G)$.

Definition 2.4. A set $F \subseteq E(G)$ is called an **edge cover** of G if every vertex of G is incident to an edge of F . We write $\rho(G)$ for the minimum size of an edge cover of G .

Lemma 2.2

For every graph G , $\alpha(G) + \tau(G) = |V(G)|$

Proof

For any vertex cover W of G , the set $V(G) \setminus W$ is independent and vice versa. Hence, W is minimum if and only if $V(G) \setminus W$ is a maximum independent set.



□

Lemma 2.3 (Gallai Lemma)

For any graph G with no isolate vertices, $\nu(G) + \rho(G) = |V(G)|$.

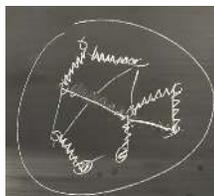
Proof

Set $|V(G)| = n$. Let M be a matching with $|M| = \nu(G)$. Then $V(G) \setminus V(M)$ is independent in G . Construct an edge cover of G as follows:

- for each $x \notin V(M)$, take an edge incident to x . Add M to this set. This gives $\rho(G) \leq |M| + (n - |V(M)|) = |M| + (n - 2|M|) = n - |M| = n - \nu(G)$.

Hence, $\nu(G) + \rho(G) \leq n$.

- Let F be a minimum edge cover of G (i.e. every vertex of G is incident to an edge of F). Let H denote the graph with $V(H) = V(G)$ and $E(H) = F$. Then each edge of H is incident to a vertex of degree 1 in H , by minimality of F . So H has no cycles hence it is a forest.



From Math239, we know $E(T) = V(T) - 1$ for any tree. For forests, $E(H) = V(H) - c(H)$ where $c(H)$ is the number of components of T . Hence, H has $V(H) - E(H) = n - \rho(G)$ components, each of which has ≥ 2 vertices (and ≥ 1 edge). By taking one edge from each component of H we get a matching of size $n - \rho(G)$ in G .

Thus, $\nu(G) \geq n - \rho(G)$

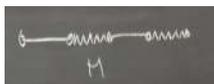
Therefore, $\nu(G) + \rho(G) = n$.

□

2.3 Alternating and Augmenting Path

Definition 2.5. Let M be a matching in a graph G .

An **M-alternating path** in G is a path in G , in which every second edge is in M .

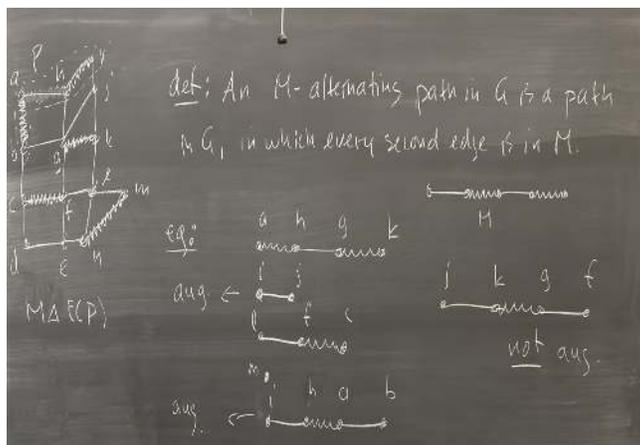


An **M-augmenting path** in G is an M-alternating path that has length at least 1 (i.e. contains at least one edge) and begins and ends with an M-exposed vertex.



If G contains an M -augmenting path P , then $M \Delta E(P) = (M \setminus E(P)) \cup (E(P) \setminus M)$ is a matching in G that is larger than M . So M is not maximum in G . We call this operation “switching” on P .

Example



2.3.1 Two matching in a graph

Suppose M_1 and M_2 are two matching in a graph G . Let H be the graph with vertex set $V(G)$ and edge set $M_1 \cup M_2$.

Then

- H has maximum degree ≥ 2 .
- Every component of H is a path or a cycle.
- Every cycle component is even, and alternates between M_1 and M_2 edges.
- Every path component either alternates $M_1 M_2$ and the number of M_1 edges differs from the number of M_2 edges by at most 1, or is a single edge in $M_1 \cap M_2$.

2.3.2 Berge's Theorem and Erdős-Pośa Theorem

Theorem 2.5 (Berge's Theorem)

A matching M in a graph G is maximal if and only if there is no augmenting path in G .

Proof

(\Rightarrow)

noted above.

(\Leftarrow)

Assume there is no M -augmenting path in G . Let M' be a maximum matching in G . If $|M'| = |M|$, then M is maximum, so suppose $|M'| > |M|$. Let H be the subgraph with $V(H) = V(G)$ and $E(H) = M \cup M'$. Then by the properties of H , there must be a component of H that is a path P of odd length that begins and ends with an M' edge.



By definition of H , the endpoints of P must be m -exposed, then P is an m -augmenting path in G , contradicting our assumption.

Therefore, M is maximum.

□

Theorem 2.6 (Erdős-Pośa)

For any graph G ,

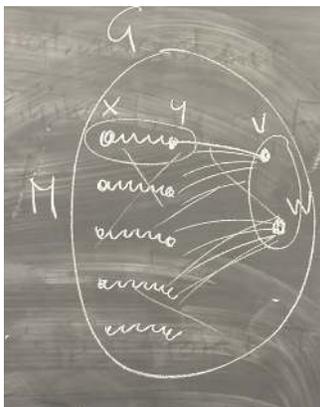
$$\nu(G) \geq \min \left\{ \left\lfloor \frac{|V(G)|}{2} \right\rfloor, \delta(G) \right\}$$

Proof

Let M be a maximum matching in G . If $|M| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, then we are done. So suppose not.

Then there are at least 2 M -exposed vertices in G , say v and w . Then all neighbours of v and

w are in the set $V(M)$ of M -saturated vertices. (since M is maximum).



For each edge $xy \in M$, if $vy \in E(G)$ then $wx \notin E(G)$, otherwise $vxyw$ is an M -augmenting path, contradicting Berge's Theorem. Similarly, $vx \in E(G) \implies wy \notin E(G)$. So the number of edges joining $\{v, w\}$ is ≤ 2 . This holds for every edge of M , hence $d(v) + d(w) \leq 2|M|$.

$$\text{So } 2\delta(G) \leq d(v) + d(w) \leq 2|M| \implies |M| \geq \delta(G).$$

□

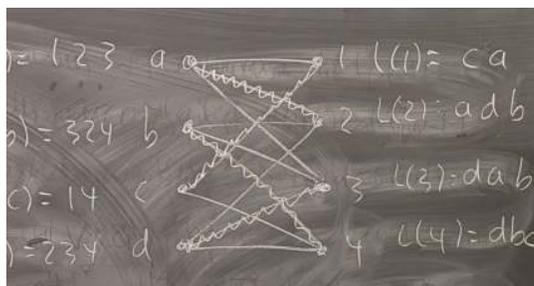
2.4 Stable Matching

Let G be a bipartite graph with vertex classes X and Y .

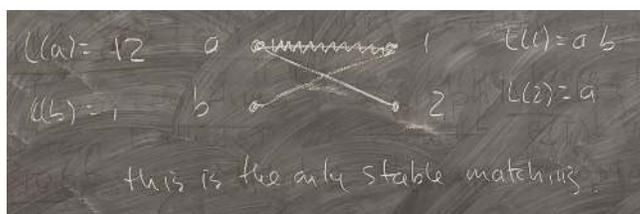
Definition 2.6. A set L of **preferences lists** for G consists of a linear order $L(z)$ of $N(z)$ for each vertex $z \in X \cup Y$.

Definition 2.7. In a bipartite graph G with preference Lists L , a matching M is **stable** if for every edge $xy \notin M$, either

- $xy' \in M$ for some y' that x prefers to y ($y' > y$ in $L(x)$), or
- $x'y \in M$ for some x' that y prefers to x ($x' > x$ in $L(y)$).



Stable matching's might not be maximum matching.



2.4.1 Gale-Shapley Algorithm

Algorithm 2: Gale-Shapley Algorithm

Input: Bipartite graph G , vertex classes X and Y , preference lists L .

Output: A stable matching M in G .

Set $k(x) := L(x)$ for all $x \in X$;

$M \leftarrow \emptyset$;

while there exists an exposed vertex $x \in X$ with $k(x) \neq \emptyset$ **do**

 Let y be the first vertex on $k(x)$;

if y is not matched **then**

 Match x and y ;

else

 Let x' be the vertex matched with y ;

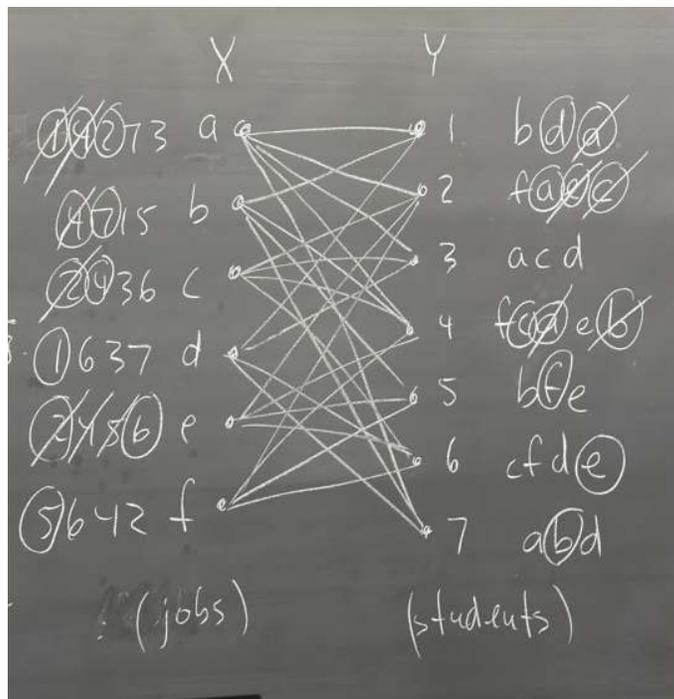
if y prefers x to x' **then**

 Unmatch x' and y ;

 Match x and y ;

 Set $k(x) := k(x) \setminus \{y\}$;

return M ;



This algorithm terminates because

$$\sum_{x \in X} k(x) \text{ start at value } \sum_{x \in X} L(x) \text{ and decreases by 1 in each iteration.}$$

Proof

The if-else statements ensure that M^* is a matching. Observe that at each iteration, the situation (†) holds:

- The situation improves or stays the same for every $y \in Y$, and
- The situation deteriorates or stays the same for every $x \in X$.

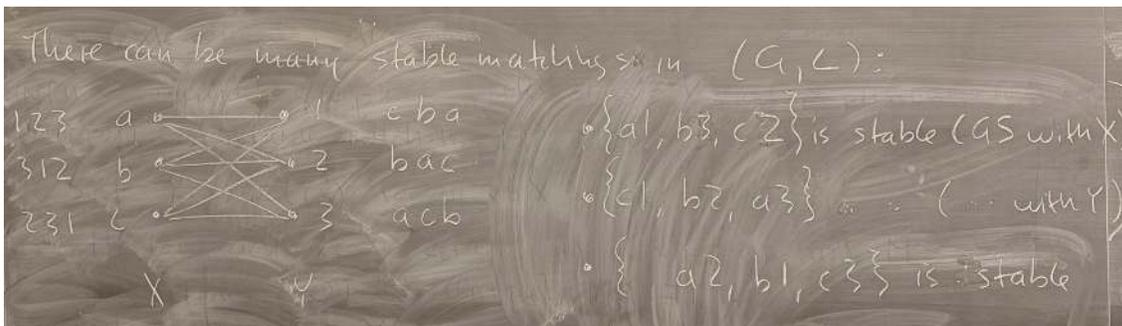
Consider an edge $x_0y_0 \notin M^*$. Then $y_0 \in L(x_0)$ and hence in $k(x_0)$ at initialization.

Case 1: x_0 proposes to y_0 in some iteration; then by (†), y_0 is matched in the final matching to some x' that y_0 prefers to x_0 .

Case 2: x_0 never proposes to y_0 ; then the algorithm terminated before x_0 reached y_0 on its list, i.e., it is matched in M^* to some y' that it prefers to y_0 .

Hence, M^* is stable.

□



Theorem 2.7

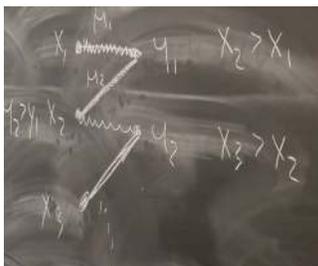
Let G be a bipartite graph with vertex classes X and Y , preference lists L . Then all stable matching in (G, L) saturate the same set of vertices.

Proof

Let M_1 and M_2 be distinct stable matching in (G, L) . Let H be the subgraph of G with $V(H) = V(G)$ and $E(H) = M_1 \cup M_2$. If M_1 and M_2 do not saturate the same set of vertices, then some component of H is an alternating path, say $x_1, y_1, x_2, y_2, \dots$ where x_1 is M_2 -exposed and $x_1 y_1 \in M_1$.

Then since $x_1 y_1 \notin M_2$, M_2 is stable, y_1 prefers x_2 to x_1 , where $x_2 y_1 \in M_2$. M_2 stable, y_1 prefers x_2 to x_1 , where $x_2 y_1 \in M_2$. $x_2 y_1 \notin M_1 \implies x_2$ prefers y_2 to y_1 .

- $x_i y_{i-1} \notin M_1 \implies x_i$ prefers y_i to y_{i-1} .
- $x_i y_i \notin M_2 \implies y_i$ prefers x_{i+1} to x_i .



This shows that no x_i can be M_1 -exposed and no y_i can be M_2 -exposed. So the path never ends, contradiction the finite graph we are working with.

Hence, M_1 and M_2 saturate the same set of vertices.

□

Corollary

All stable matching in (G, L) are the same size.

Definition 2.8. Let G be a bipartite graph with vertex classes X and Y , preference lists L . A stable matching M_0 in G is **X-optimal** if for every stable matching M in (G, L) , and every $x \in X$,

- if $xy \in M$ then there exists y' such that $xy' \in M_0$ and $y' \geq y$ in $L(x)$ (i.e. either $y' = y$ or x prefers y' to y).

Similarly, M_0 is **X-pessimal** means for every $x \in X$,

- If $xy \in M$ then there exists y' with $xy' \in M_0$ and $y' \leq y$ in $L(x)$ (i.e. either $y' = y$ or x prefers y to y').

Note

The X-optimal (X-pessimal) matching is unique, if it exists.

We'll show that the G-S stable matching with X , making the proposals is X-optimal for every bipartite graph $G = (X, Y)$ with strict preference lists.

Proof

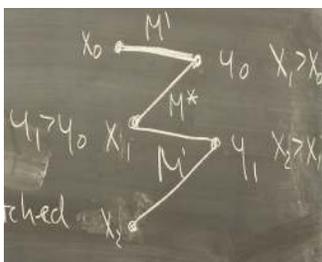
Suppose on the contrary, that there exists a stable matching M' in G , and an edge $x_0, y_0 \in M'$ (where $x_0y_0 \notin M^*$) such that x_0 prefers y_0 to its partner in M^* .

Since $x_0y_0 \notin M^*$, y_0 is matched to some x_1 , where $x_1 > x_0$ in $L(y_0)$.



Let I be the earliest iteration in the G-S algorithm in which we put an edge x_1y_0 into the matching M , where $x_0, y_0 \in M'$ and x_0 strictly prefers y_0 to its partner in M^* .

Since $x_1y_0 \notin M'$, and y_0 prefers x_1 to x_0 , we know x_1 is matched in M' to some y_1 , that it prefers to y_0 .



In iteration I , x_1 prefers to y_0 , and it accepted. Since x_1 prefers y_1 to y_0 , in some earlier iteration I_1 , x_1 was rejected by y_1 , since in some earlier iteration I_1 , y_1 received and accepted an offer from some x_2 , where $x_2 > x_1$ in $L(y_1)$.

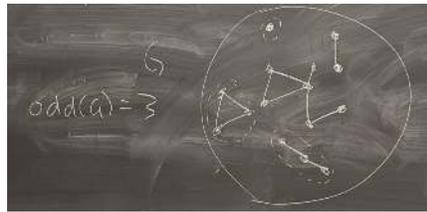
So in I_1 , x_2y_1 was put into M , where $x_1y_1 \in M'$, and x_1 strictly prefers y_1 to its partner in M^* . This contradicts the choice of iteration I , so x_0y_0 does not exist.

M^* is **X-optimal**.

□

2.5 Perfect matching in general graphs

Definition 2.9. A **odd** component of a graph G is a component of G with an odd number of vertices. The parameter $odd(G)$ is the number of odd components in G .

**Note**

Suppose a graph G has a perfect matching M . Then for any subset $T \subseteq V(G)$, there is an edge of M from each odd component of $G - T$ to T .



Hence, in particular $|T| \geq \text{odd}(G - T)$.

So if G has a perfect matching, then for every $T \subseteq V(G)$, $|T| \geq \text{odd}(G - T)$.

2.5.1 Tutte's Theorem

Lemma 2.4 (A)

Let G be a graph that satisfies

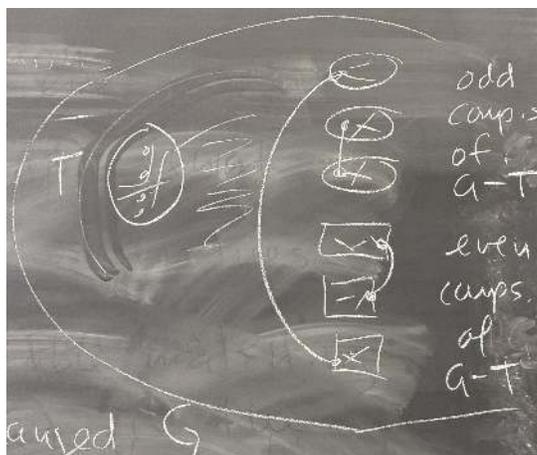
- $\text{odd}(G - T) \leq |T|$ for every $T \subseteq V(G)$

If H is a graph with $V(H) = V(G)$ and $E(H) \supseteq E(G)$, then H also satisfies

Proof

Let $T \subseteq V(H) = V(G)$.

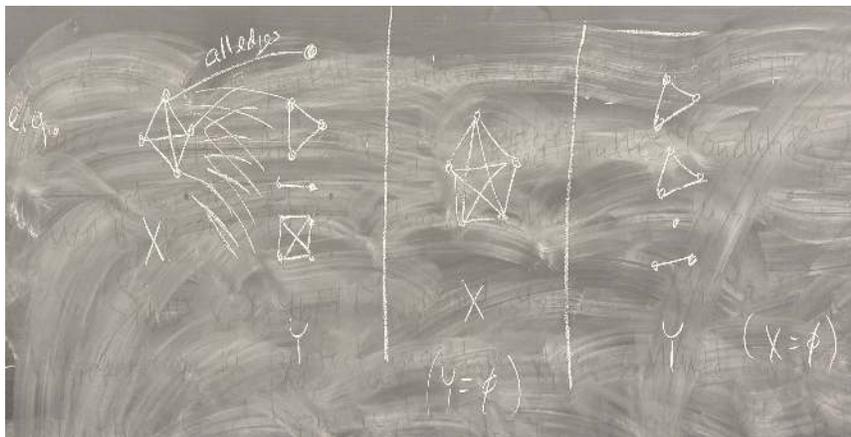
- Adding an edge inside T , or inside a component of $G - T$, or crossing from T to $G - T$ does not change $\text{odd}(G - T)$.
- Adding an edge between two components:
 - Between two even components: does not change $\text{odd}(G - T)$.
 - Between two odd components: decreases $\text{odd}(G - T)$ by 2.
 - Between an odd component and an even component: decreases $\text{odd}(G - T)$ by 1.



Hence, $\text{odd}(H - T) \leq \text{odd}(G - T)$ for every T , which implies the statement.

□

Definition 2.10. A graph G is **Type-0** if $V(G)$ has a partition $X \cup Y$ where $X = \{x \in V(G) : xy \in E(G) \text{ for all } y \neq x \in V(G)\}$, and every component of $Y = V(G) - X$ is a complete graph.



Lemma 2.5 (B)

If G is a type-0 graph with Tutte's condition, then G has a perfect matching.

Proof

When $T = \emptyset$, we find $odd(G) = 0$, so $|V(G)|$ is even.

When $T = X$, we find $odd(G - X) \leq |X|$. So there are $\leq |X|$ odd complete components in $G - X$, we find $odd(G - X) \leq |X|$.

So there are $\leq |X|$ odd complete components in $G - X$. Form a matching M by matching one vertex from each odd component of $G - X$ to a vertex in X . Match all remaining vertices in each component C of $G - X$ within C . Match any remaining vertices in X within X ,

□

Theorem 2.8 (Tutte's Theorem)

A graph G has a perfect matching if and only if for every $T \subseteq V(G)$, $|T| \geq odd(G - T)$.

$$\forall T \subseteq V(G), |T| \geq odd(G - T) \quad (\text{called the Tutte condition})$$

If G has a perfect matching then $|V(G)|$ is even. Tutte's condition with $T = \emptyset$ implies $\text{odd}(G) = 0$, so every component of G is even, hence $|V(G)|$ is even.

Proof

(\Rightarrow): Obvious from earlier note.

(\Leftarrow):

Let G has Tutte's condition. Suppose on the contrary that G has no perfect matching. Let H be a graph such that

- $V(H) = V(G)$
- $E(H) \subseteq E(G)$
- H has no perfect matching
- (\dagger) $H \cup \{ab\}$ for any $a, b \in V(H)$ where $ab \notin E(H)$ has a perfect matching.

We can form H from G by adding new edges until our last property holds. Then H satisfies Tutte's condition by Lemma (A).

Aim: to show H is type-0.

Let $X = \{x \in V(H) : xz \in E(H) \text{ for all } z \neq x \in V(H)\}$ ($x = \emptyset$ is possible), we want to show that every component of $H - X$ is complete. Suppose C is a component of $H - X$ that is not complete then $|V(C)| \geq 3$.

Claim: There exist vertices $a, b, c \in V(C)$, and $d \in V(H) \setminus \{a, b, c\}$ such that $ab, bc \in E(H)$, $ac \notin E(H)$, and $bd \notin E(H)$.

Since C is not complete we can find vertices a and y in C such that $ay \notin E(H)$. Let P be a shortest path from a to y . Let b and c be the second and third vertices on P . Then $ac \notin E(H)$ because P is the shortest path. There exists d not adjacent to b because $b \notin X$.

By (\dagger), the graph $H \cup \{ac\}$ has a perfect matching M_1 , and the graph $H \cup \{bd\}$ has a perfect matching M_2

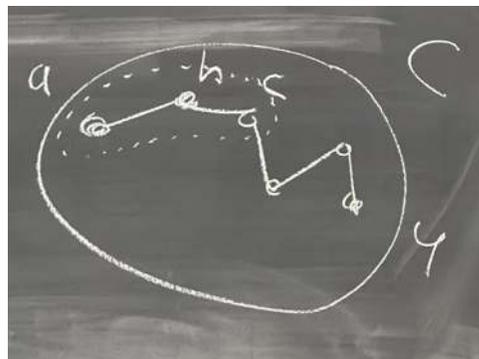
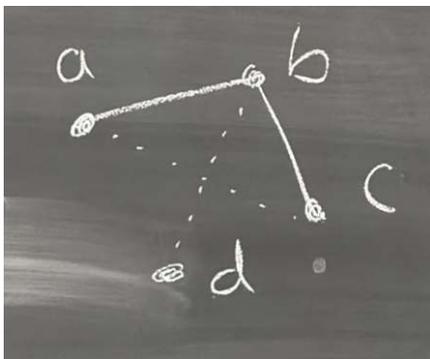
Let J be the graph with vertex set $v(H)$ and edge set $M_1 \cup M_2$. Then every component of J

is

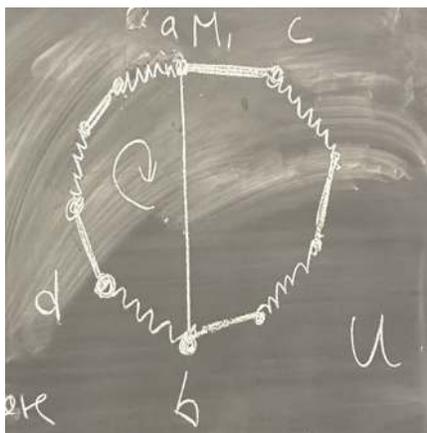
- a single edge in $M_1 \cap M_2$
- an alternating cycle (no path component, since M_1 and M_2 both saturate $V(H)$)

Then ac in an alternating cycle component U of J . If bd is not in U , then $M_1 \Delta E(U)$ is a perfect matching of H , contradiction.

So we may assume $bd \in E(U)$. By symmetry of a and c , we may assume there is a path in U from d to a that does not contain c or b .



Then the matching $M_2 \Delta U'$ is a perfect matching of H , where U' is the cycle $Q \cup \{ab, bd\}$. This contradiction shows H is type-0, hence by Lemma (B), H has a perfect matching, contradiction.



□

2.6 Applying Tutte's Theorem

Lemma 2.6

Let G be a graph with $|V(G)|$ even. Then for every $T \subseteq V(G)$, we have $odd(G - T) \equiv |T| \pmod{2}$.

Proof

Since $|V(G)| \equiv 0 \pmod{2}$, we find $|T| + |V(G - T)| = |V(G)| \equiv 0 \pmod{2}$, so $|T| \equiv |V(G - T)| \pmod{2}$.

$$\begin{aligned}
 |V(G - T)| &\equiv \sum_{C \in \text{odd components of } G-T} |V(C)| + \sum_{C \in \text{even components of } G-T} |V(C)| \\
 |V(G - T)| &\equiv \sum_{C \text{ odd}} 1 + \sum_{C \text{ even}} 0 \pmod{2} \\
 |V(G - T)| &\equiv odd(G - T) \pmod{2}
 \end{aligned}$$

□

Definition 2.11. Recall from Math 239 that a bridge in a connected graph G is an edge e such that $G - e$ is disconnected.

Theorem 2.9 (Petersen's Theorem)

Let G be a connected cubic graph with at most two bridges. Then G has a perfect matching.



Proof

Note $|V(G)|$ is even since it is cubic (Handshaking Lemma).

Suppose that G has no perfect matching.

Then by Tutte's Theorem, there exists $T \subseteq V(G)$ with $\text{odd}(G - T) > |T|$.

By the previous Lemma, $\text{odd}(G - T) \geq |T| + 2$.

Claim: For each odd component C of $G - T$, the number m_C of edges joining C to T is odd.

$$3|V(C)| = \sum_{v \in V(C)} d_G(v) = 2|E(C)| + m_C$$

So m_C is odd since $3|V(C)|$ is odd and $2|E(C)|$ is even.

For at most two components C we have $m_C = 1$, because such a single edge is a bridge of G .

So the number of edges joining odd components of $G - T$ to T is

$$\geq 3(\text{odd}(G - T) - 2) + 2 \geq 3(|T| + 2 - 2) + 2 = 3|T| + 2$$

But since the graph is cubic, this is not possible.

□

Theorem 2.10 (Defect version of Tutte's Theorem)

Let G be a graph and let $d = \max_{T \subseteq V(G)} \{\text{odd}(G - T) - |T|\}$.

Then G has a matching that saturates at least $|V(G)| - d$ vertices.

Proof

Note $d \geq 0$ since $\text{odd}(G - \emptyset) - |\emptyset| = \text{odd}(G) \geq 0$.

If $d = 0$, then this is Tutte's theorem. So we may assume $d \geq 1$. By definition of d , for every $T \subseteq V(G)$, $\text{odd}(G - T) \leq |T| + d$.

Contract a new graph H by adding a set A of d new vertices, and adding edges $\{ax : x \in V(H) \setminus \{a\}\}$ for each $a \in A$.

- $|V(H)|$ is even: for some $T_0 \subseteq V(G)$, $d = \text{odd}(G - T_0) - |T_0|$. So, $|V(H)| = |V(G - T_0)| + |T_0| + d \equiv \text{odd}(G - T_0) - |T_0| + d \equiv 2d \equiv 0 \pmod{2}$.
- Let $\emptyset \neq S \subseteq V(H)$ where $A \not\subseteq S$. Then $\text{odd}(H - S) \leq 1$. Since $H - S$ is connected. Hence, $\text{odd}(H - S) \leq 1 \leq |S|$.
- Let $S = \emptyset$. Then $\text{odd}(H - S) = \text{odd}(H) = 0$, since H is connected and $|V(H)|$ is even.
- $S \subseteq V(H)$ where $A \subseteq S$. Then $\text{odd}(H - S) = \text{odd}(G - (S \setminus A)) \leq |S \setminus A| + d = |S| - d + d = |S|$.

Hence, by Tutte's theorem, H has a perfect matching M .

At most d vertices of G can be incident to edges of M that are not edges of G (i.e. their other endpoint is in A). so M saturates at least $|V(G)| - d$ vertices of G .

□

2.6.1 The Tutte-Berge Formula

Theorem 2.11

Let G be a graph and let $d = \max_{T \subseteq V(G)} \{\text{odd}(G - T) - |T|\}$. Then

$$\nu(G) = \frac{|V(G)| - d}{2}$$

Proof

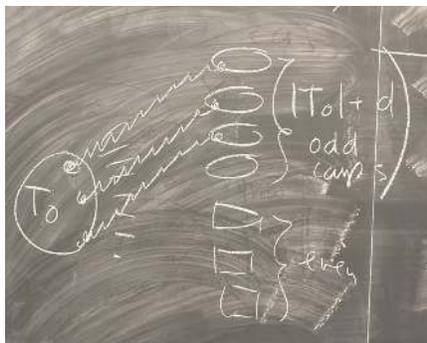
Recall from the previous proof that

$$|V(G)| - d \equiv |V(G)| + d \equiv 0 \pmod{2}$$

By the defect version of Tutte's theorem we know

$$\nu(G) \geq \frac{|V(G)| - d}{2}$$

Let T_0 be such that $d = \text{odd}(G - T_0) - |T_0|$. Any matching in G can have an edge from at most $|T_0|$ odd components to T_0



The remaining d odd components will each have an M-exposed vertex. So

$$\nu(G) \leq \frac{|V(G)| - d}{2}$$

Hence, $\nu(G) = \frac{|V(G)| - d}{2}$.

□

Edmonds' Matching Algorithm finds a maximum matching in a general graph. It is efficient.

Tip

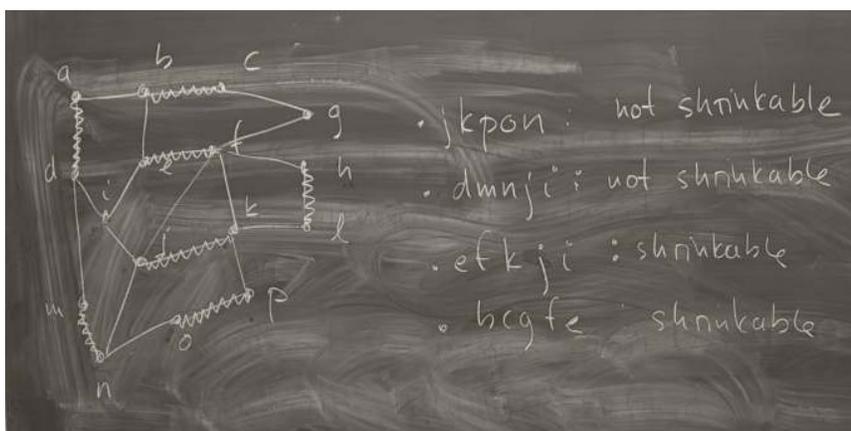
What to do with odd cycles?

2.7 Cycle Shrinking

Definition 2.12. Let G be a graph and let M be a matching in G . An odd cycle C of length $= 2k + 1$ is shrinkable (with respect to M) if

- $|M \cap E(C)| = k$
- C contains an M -exposed vertex

Example

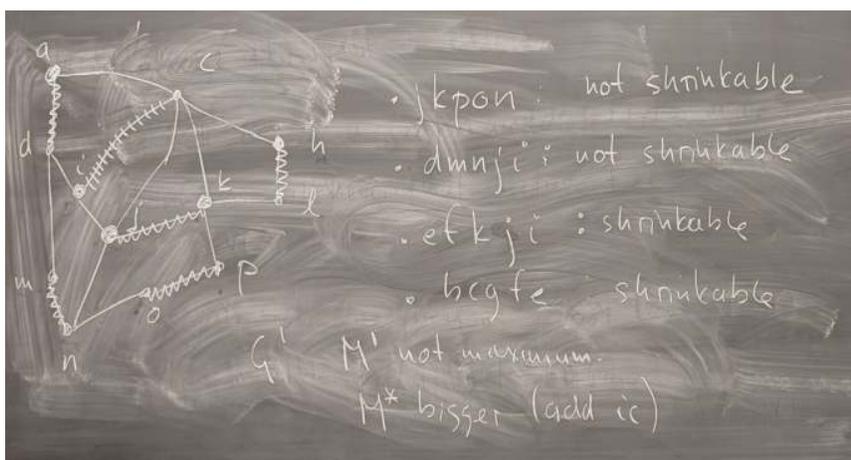
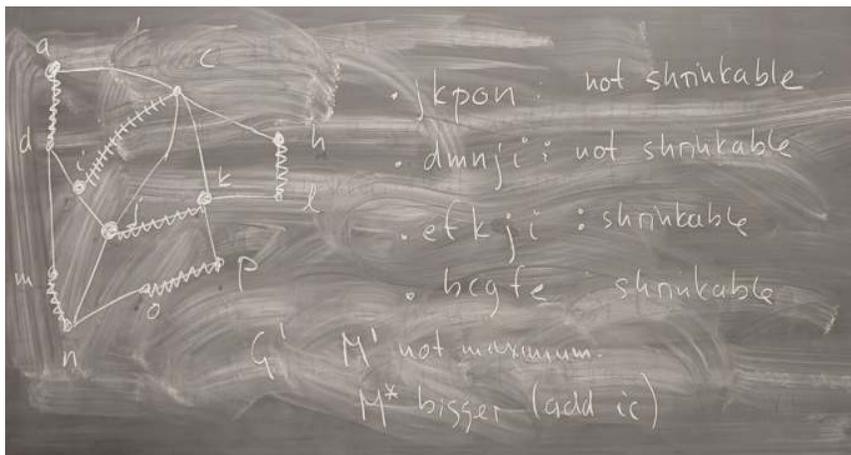


Lemma 2.7 (The cycle shrinkable lemma)

Let M be a matching in a graph G . Suppose C is a shrinkable odd cycle with respect to M in G . Let G' be the graph obtained from G by contracting every edge of C .

(the operation of “cycle shrinking”). Then M is maximum in G if and only if the matching $M' = M \setminus E(C)$ is maximum in G' .

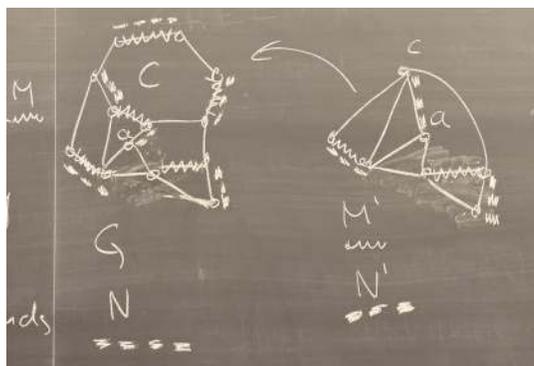
Example



Proof

(\Rightarrow)

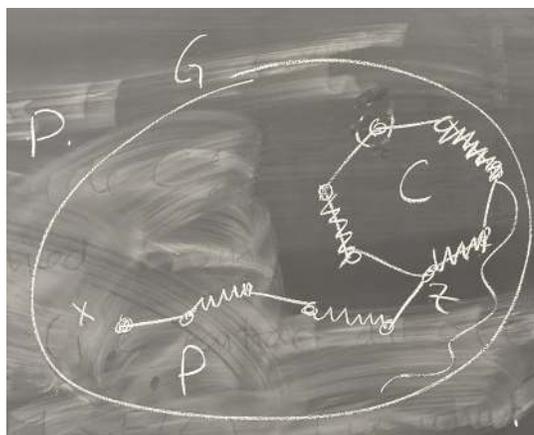
Assume M is maximum in G . Suppose on the contrary that M' is not maximum in G' . Then there exists a larger matching N' in G' , i.e., $|N'| > |M'|$ in G' , then N' corresponds to a matching in G that saturates at most one vertex of C . Then we can add k edges of C (where C has length $2k + 1$) to set a matching N in M that is bigger than M , a contradiction.



(\Leftarrow)

Now suppose M' is maximum in G' . If M is not maximum in G then by Berge's theorem there is an M -augmenting path P in G . If P is disjoint from C , then P is also an M' -augmenting path in G' , contradicting that M' is maximum in G' . So we may assume P contains vertices of C .

Let x be an endpoint of P that is not on C (which exists since both endpoints of P are M -exposed).

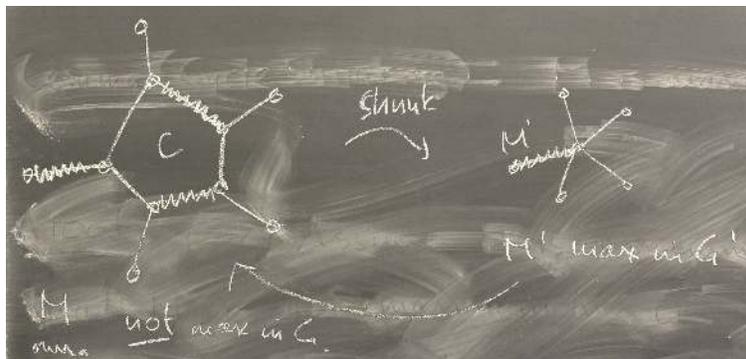


Let z denote the vertex of C closest to x along P . Then the (x, z) -segment of P is an M' -augmenting path in G' contradicting that M' is maximum in G' .

□

Note

Why is the M -exposed vertex on C necessary?



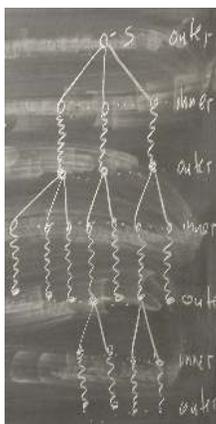
In the proof if N' is maximum in G' , and $|N'| > |M'|$, it may not happen that N is maximum in G (It is bigger than M but may not be maximum).

Definition 2.13. Let G be a graph and let M be a matching in G . A **M -alternating tree** in G is a subgraph T of G such that

- T is a tree,
- T contains exactly one M -exposed vertex, called the **root** of T ,
- Each edge of T at an odd distance from s (the root) in T is an edge of M .
- Each vertex of T at an odd distance from s in T has degree exactly 2 in T .

These are called the **inner vertices** of T .

All the other vertices of T are called the **outer vertices** of T .



Definition 2.14. An ***M*-alternating forest** in G is a subgraph F of G such that every component of F is an M -alternating tree.

- The set $I(F)$ of inner vertices of F is

$$I(F) = \bigcup_{T \in \text{components of } F} I(T)$$

- Similarly, outer vertices of F ,

$$O(F) = \bigcup_{T \in \text{components of } F} O(T)$$

A maximal M -alternating forest is one that is not contained in a strictly larger M -alternating forest.



Lemma 2.8

Let M be a matching in G and let F be a maximal M -alternating forest in G . If there is no edge of G joining two outer vertices of F , then M is a maximum matching in G .

Proof**Note**

- (i) Every M -exposed vertex in G is in F (by maximality of F).
- (ii) If $xy \in M$ and x is in a component T of F , then y is also in T .

Claim: If $z \in O(F)$ then $N_G(z) \subseteq I(F)$.

Let $y \in N_G(z)$. By assumption, $y \notin O(F)$. Suppose $y \notin V(F)$. Then y is saturated by M , by (i) above, its partner in M is in F . Let x be such that $xy \in M$. Then $x \notin V(F)$ by (ii) above. But then $F \cup \{zy, yx\}$ is a larger M -alternating forest. So $N_G(O(F)) \subseteq I(F)$.



For every component T of F , $O(T) - I(T) = 1$ since M pairs up all outer vertices with inner vertices except the root. Thus, $O(F) - I(F) = \text{number of components of } F := k$ (say).

Consider the graph $G - I(F)$. In this graph, each $z \in O(F)$ is an isolated vertex, i.e. a component of size 1. Therefore $\text{odd}(G - I(F)) \geq |O(F)| = |I(F)| + k$.

The Tutte-Berge formula says $\nu(G) = \frac{|V(G)| - d}{2}$ where

$$d = \max_{S \subseteq V(G)} (\text{odd}(G - S) - |S|)$$

So $d \geq \text{odd}(G - I(F)) - |I(F)| \geq k$.

Therefore, $\nu(G) = \frac{|V(G)| - d}{2} \leq \frac{|V(G)| - k}{2} = |M|$.

Thus, M is maximum.

□

Recall

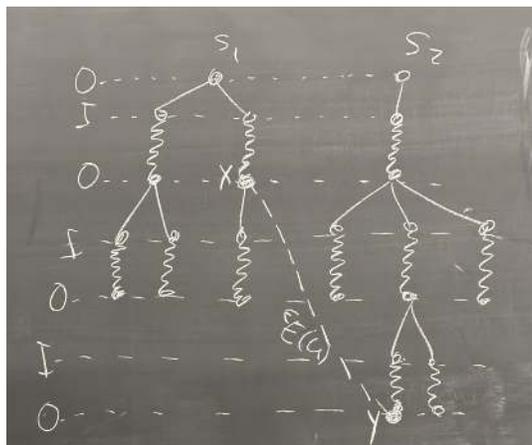
- Maximal M -alternating forest F in G .
- If there is no edge of G joining two outer vertices of F , then M is a maximum matching in G .

Lemma 2.9

Let M be a matching in G_1 and let T_1 and T_2 be distinct components of an H -alternating forest F in G . Suppose $xy \in E(G)$ where $x \in O(T_1)$ and $y \in O(T_2)$. Then G contains an M -augmenting path.

Proof

Since the roots s_1 and s_2 of T_1 and T_2 respectively are both M -exposed, The unique path $T_1 \cup T_2 \cup \{xy\}$ is an M -augmenting path. (Also use the definition of M -alternating forest.)



□

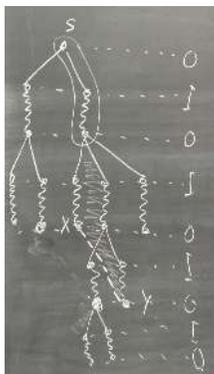
Lemma 2.10

Let M be a matching in G . Let T be a component of an M -alternating forest F in G . Suppose there exists an edge $xy \in E(G)$ where $x, y \in O(T)$. Then there exists a matching \bar{M} in G with $|\bar{M}| = |M|$ and a shrinkable odd cycle with respect to \bar{M} in G .

Proof

Let C be the unique cycle in $T \cup \{xy\}$. Let P be the unique path from C to the root s of T in T . Set $\bar{M} = M \Delta E(P)$. (“switch M on P ”)

Then $|\bar{M}| = |M|$ since P is an even length M -alternating path and C is shrinkable respect to \bar{M} .



□

2.7.1 Edwards Algorithm for maximum matching**Algorithm 3:** Edwards Algorithm for Maximum Matching**Input:** Graph G **Output:** Maximum Matching M in G $M \leftarrow \emptyset;$ **while** there exists an M -augmenting path or a shrinkable odd cycle in G **do** Construct a maximal M -alternating forest F in G ; **if** there exists an edge $xy \in E(G)$ joining two outer vertices of F **then** **if** x and y are in distinct components of F **then** Augment M along the unique M -augmenting path in $F \cup \{xy\}$;

(By Lemma 2.9)

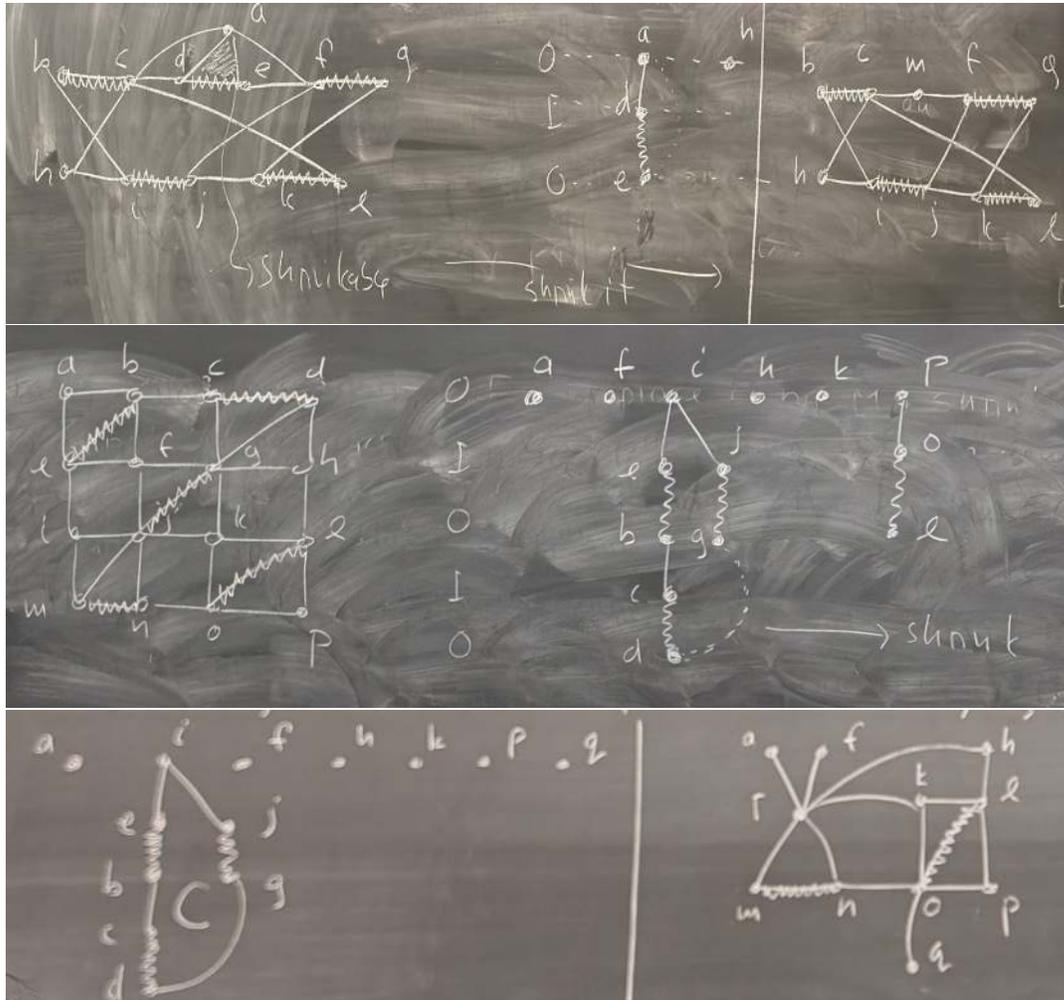
else Shrink the unique shrinkable odd cycle in $F \cup \{xy\}$;

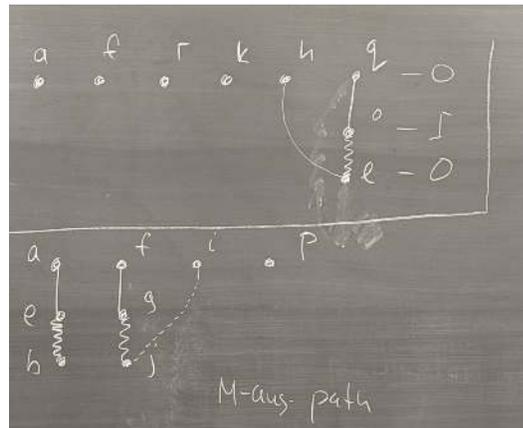
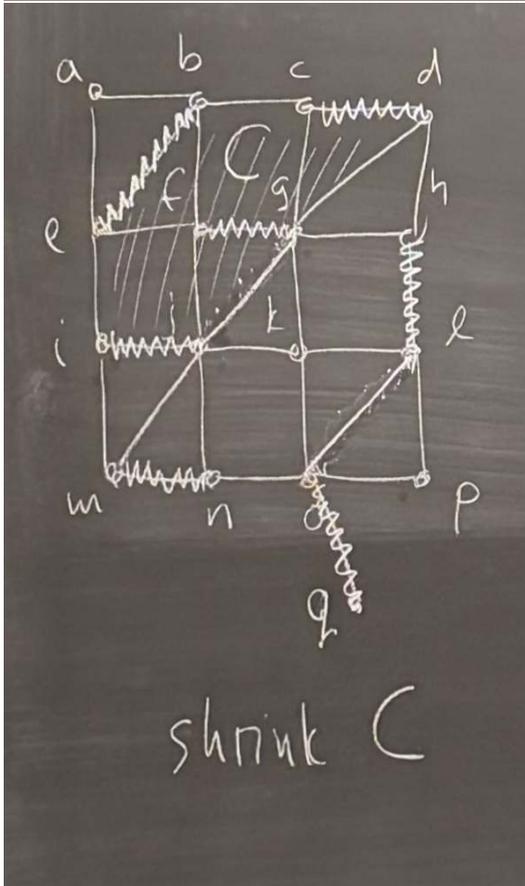
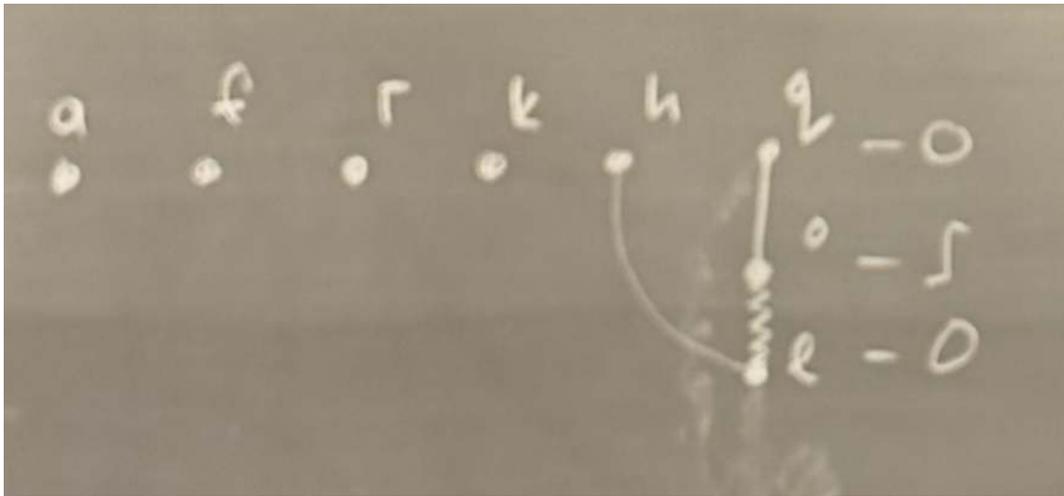
(By Lemma 2.10)

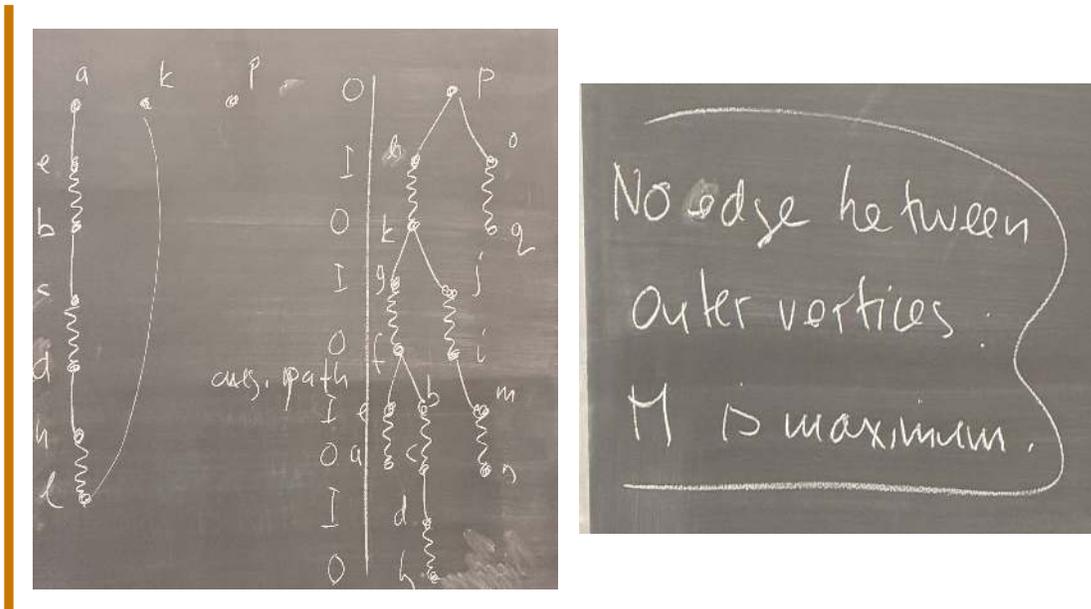
return M ;

Example

Running Edmonds Algorithm:







2.8 f-factors

Definition 2.15. Let G be a graph and let $f : V(G) \rightarrow \mathbb{N}$ be a function. An **f-factor** in G is a spanning subgraph H (i.e., $V(H) = V(G)$) such that for every $v \in V(G)$, $d_H(v) = f(v)$.

For example, a perfect matching in G is a 1-factor in G .

Where $\mathbb{1}$ is the function defined by $\mathbb{1}(v) = 1$ for each $v \in V(G)$,

Q: Given G and f , does G contain an f -factor?

If for some v , $f(v) > d_G(v)$, then clearly no f -factor exists.

The graph $H(G, f)$, given a graph G and a function $f : V(G) \rightarrow \mathbb{N}$, such that $f(v) \leq d_G(v)$ for each v , the graph $H(G, f)$ has vertex set

$$\bigcup_{v \in V(G)} A(v) \cup B(v)$$

where

$$A(v) = \{v_1, v_2, \dots, v_{d_G(v)}\}, \quad |A(v)| = d_G(v) \geq 0$$

and

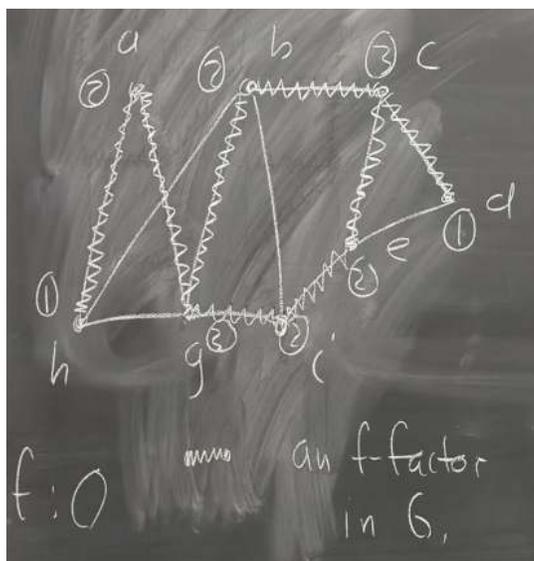
$$B(v) = \{v'_1, v'_2, \dots, v'_{d_G(v)-f(v)}\}, \quad |B(v)| = d_G(v) - f(v) \geq 0$$

The edge set of $H(G, f)$ is defined as follows:

- For each $u, v \in V(G)$ such that $uv \in E(G)$, add the edges $u_i v_j$ for all $i \in [d_G(u)]$ and $j \in [d_G(v)]$.
- For each $v \in V(G)$, add the edges $v_i v'_j$ for all $i \in [d_G(v)]$ and $j \in [d_G(v) - f(v)]$.

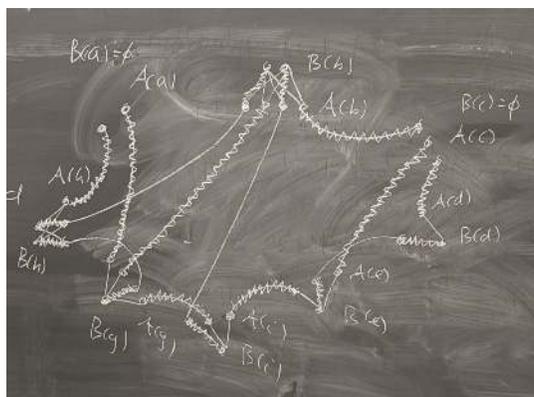
$H(G, f)$ has exactly one edge e_{vw} for each edge $vw \in E(G)$, which joins a vertex in $A(v)$ to a vertex in $A(w)$, such that $\{e_{vw} : vw \in E(G)\}$ forms a matching in $H(G, f)$. In addition, $H(G, f)$ has all edges joining $A(v)$ to $B(v)$ for each $v \in V(G)$.

Example



Theorem 2.12 (Tutte's f-factor Theorem)

Let G be a graph and let $f : V(G) \rightarrow \mathbb{N}$ be such that $f(v) \leq d_G(v)$ for each v . Then G has an f -factor if and only if $H(G, f)$ has a perfect matching.



Proof

(\Rightarrow) Suppose J is an f -factor in G . Let M be the matching in $H(G, f)$ consisting of $\{e_{vw} : vw \in E(J)\}$. For each $v \in V(G)$, add to M , $d_G(v) - f(v)$ edges joining $A(v)$ to $B(v)$ that are disjoint from M . This is possible since $|A(v)| = |B(v)| + f(v)$. This forms a perfect matching in $H(G, f)$.

(\Leftarrow) Suppose $H(G, f)$ has a perfect matching M . Then each $B(v)$ is saturated by M , and $N(B(v)) = A(v)$. So $B(v)$ saturates by M to exactly $|B(v)|$ vertices of $A(v)$.

Since $A(v)$ is also saturated, there must be exactly $|A(v)| - |B(v)| = f(v)$ edges of H of the form e_{vw} for same $w \in V(G)$. So all these edges together correspond to edges vw of an f -factor in G .

□

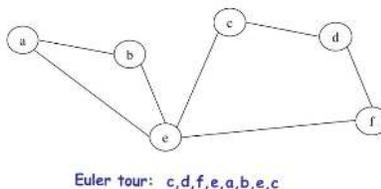
2.9 Even graphs

Definition 2.16. A graph G is said to be even if $d(v)$ is even for each $v \in V(G)$.

Definition 2.17. An **Euler tour** of a graph G is a sequence of $v_0, e_1, v_1, e_2, \dots, e_k, v_k$ of vertices and edges of G such that

- $e_i = v_{i-1}v_i$ for each i
- $v_k = v_0$
- $e_i \neq e_j$ for $i \neq j$
- $k = |E(G)|$ (i.e., every edge of G appears exactly once)

Example



Theorem 2.13 (from Math 239)

If G is connected and even then G has an Euler tour.

Definition 2.18. For a positive integer r , an **r -factor** in a graph G is an f -factor for the function

$$f : V(G) \rightarrow \{1, 2, \dots\}$$

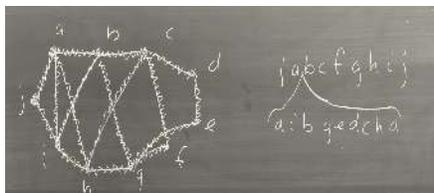
defined by $f(v) = r$ for all $v \in V(G)$.

Theorem 2.14 Petersen’s 2-factor theorem

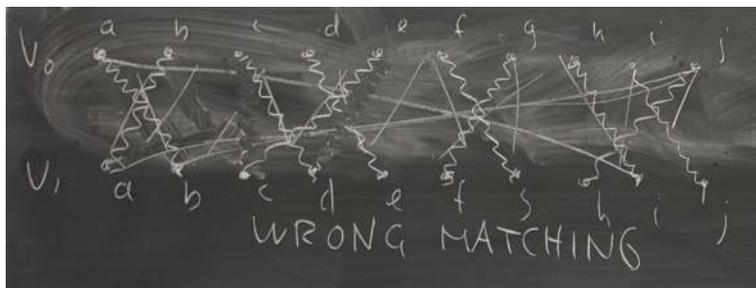
Let $d \geq 2$ be even. Then every d -regular graph contains a 2-factor.

Proof

We may assume G is connected, otherwise we can consider all its components separately.



Then G has an Euler tour Q . Define the bipartite graph B as follows: $V(B) = V_0 \cup V_1$ where V_0 and V_1 are disjoint copies of $V(G)$.



The edge set is

$$\{v_0w_1 : vw \in E(G), v_0 \text{ is the copy of } v \text{ in } V_0 \text{ and } w_1 \text{ copy of } w \text{ in } V_1 \text{ and } vw \text{ is traversed in the direction } v \text{ to } w \text{ in } Q \}$$

Then every vertex of B is $\frac{d}{2}$, since each $v \in V$ has exactly $\frac{d}{2}$ edges in Q (which is $d_B(v_0)$) and exactly $\frac{d}{2}$ edges out of Q (which is $d_B(v_1)$).

Then B has a perfect matching M (since it is regular of positive degree).

Therefore M consists of exactly one in-edge and one out-edge for every vertex $v \in V(G)$. So M defined a 2-regular spanning subgraph of G , i.e., a 2-factor in G .

□

Vector Spaces associated with Graphs

Recall

The principal properties of a vector space W over field \mathbb{F} are:

- W contains a zero vector $\vec{0}$
- any two $v, w \in W$ can be added to set some $v + w \in W$
- any $v \in W$ can be multiplied by any $c \in \mathbb{F}$ to get some $cv \in W$

If $u \subseteq W$ and u is also a vector space over \mathbb{F} , we say u is a **subspace** of W .

Definition 3.1. Let G be a graph, and let S be a spanning subgraph of G . (i.e. $V(S) = V(G)$). The **characteristic vector** w_s of S is the $(0,1)$ -vector with coordinates indexed by $E(G)$ defined by w_s i^{th} coordinate is 1 if $e_i \in E(S)$ and 0 otherwise.

3.1 The Flow space

Definition 3.2. Let G be a graph. The **flow space** $W(G)$ of G is the set of all characteristic vectors of even spanning subgraphs of G . This is a vector space over the field \mathbb{Z}_2 :

- $W(G)$ contains a zero vector because the spanning subgraph of G with 0 edges is even.
- Addition in $W(G)$: $w_s + w_t$ is the characteristic vector of the spanning subgraph $S \oplus T$

which has edges set $E(S) \cup E(T) \setminus E(S) \cap E(T)$. Then $d_{S \oplus T}(v) = d_S(v) + d_T(v) - 2d_{S \cap T}(v)$, which is even.

So $w_s + w_t \in W(G)$.

- Scalar multiplication $0 \cdot v = 0$ and $1 \cdot v = v$ for each $v \in W(G)$.

Theorem 3.1

Let G be a graph and let H be an even spanning subgraph of G . Then there exist cycles C_1, C_2, \dots, C_k in G such that $E(H) = \bigcup_{i=1}^k E(C_i)$ and $E(C_i) \cap E(C_j) = \emptyset$ for $i \neq j$.

Proof

By induction on $|E(H)|$.

Base case: If $|E(H)| = 0$, then take $k = 0$, and we are done.

IH: Assume $|E(H)| > 0$ and that for any even spanning subgraph J of G with $|E(J)| < |E(H)|$, we can find a set of disjoint cycles whose union gives $E(J)$.

For some component H_1 of H , there exists a cycle in H , since $d_{H_1}(v) \geq 2$ for all $v \in V(H_1)$. Call this cycle C_1 . Then the spanning subgraph I defined by $E(I) = E(H) \setminus E(C_1)$ is also even, since

$$d_I(v) = \begin{cases} d_H(v) & v \notin V(C_1) \\ d_H(v) - 2 & v \in V(C_1) \end{cases}$$

Hence, by IH, there exist cycles C_2, C_3, \dots, C_k in I such that $E(I) = \bigcup_{i=2}^k E(C_i)$ and $E(C_i) \cap E(C_j) = \emptyset$ for $i \neq j$.

Hence, $E(H) = \bigcup_{i=1}^k E(C_i)$ and $E(C_i) \cap E(C_j) = \emptyset$ for $i \neq j$.

This implies $W(G)$ is spanned by the set of characteristic vectors of cycles in G .

□

3.2 Basis for $W(G)$

Definition 3.3. Let G be a connected graph and let T be a spanning tree in G . The unique cycle C_e in $T \cup \{e\}$ is called the **fundamental cycle** of e w.r.t T , for any $e \in E(G) \setminus E(T)$.

Theorem 3.2

Let G be a connected graph and let T be a spanning tree of G .

Then

$$B_T = \{w_C : C \in F(T)\}$$

is a basis for the flow space $W(G)$, where $F(T)$ is the set of fundamental cycles with respect to T .

Proof

First we show B_T is linearly independent.

Let $F(T) = \{C_e : e \in E(G) \setminus E(T)\}$

Suppose

$$\begin{aligned} \sum_{e \in E(G) \setminus E(T)} \lambda_e w_{C_e} &= 0 \quad (0 \text{ vector}) \\ &= (\text{characteristic vectors of the fundamental cycles}) \end{aligned}$$

for some $\lambda_e \in \{0, 1\}$ for each e .

For each $e \in E(G) \setminus E(T)$, the number of w_{C_e} that have a 1 in the e -th coordinate is exactly 1, namely w_{C_e} itself. Hence $\lambda_e = 0$

Hence, B_T is linearly independent.

B_T spans $W(G)$: Let H be an even spanning subgraph of G . **Claim:** $w_H = \sum_{e \in E(H) \setminus E(T)} w_{C_e}$.

Note this is the characteristic vector of $\bigoplus_{e \in E(H) \setminus E(T)} C_e := J_H$.

We would like $J_H = H$. i.e. $J_H \oplus H$ is the empty graph.

Each $e \in E(G) \setminus E(T)$ is an edge of J_H , since c_e is the unique fundamental cycle containing e . So $J_H \oplus H$ does not contain e .

So, $J_H \oplus H$ is a subgraph of T . If $J_H \oplus H$ has any edges, it has a vertex of degree 1. So, $J_H \oplus H$ is not even, a contradiction.

□

Corollary

For any connected graph G with n vertices and m edges, the dimension of the flow space $W(G)$ is $m - n + 1$.

Proof

Any spanning tree of G has $n - 1$ edges.

□

3.3 Binary Codes

Definition 3.4. A **binary code** of length m is a subspace U of $\mathbb{GF}(2)^m$.

Definition 3.5. The **minimum distance** of U is the smallest t such that some $0 \neq u \in U$ has exactly t 1's.

Definition 3.6. The **hamming distance** between $u, v \in U$ is the number of 1-coordinates in $u + v$.

Lemma 3.1

Let U be a binary code with minimum distance t then if $u, v \in U$ are disjoint then they are at Hamming distance at least t .

Proof

Since U is a subspace $u + v \in U$.

□

Definition 3.7. The **girth** of a graph G is the length of a shortest cycle in G . (∞ if G is a forest)

Lemma 3.2

Let G be a connected graph with finite girth g . Then the flow space $W(G)$ is a binary code of length $m = |E(G)|$, dimension $m - n + 1$ where $n = |V(G)|$, and minimum distance g .

Proof

We just need to check the minimum distance:

Every non-zero element w_H of $W(G)$ is the characteristic vector of H , an even spanning subgraph which contains a cycle C . Since $|E(C)| \geq g$, the number of 1's in w_H is at least g . (Equality holds since the shortest cycle gives an even spanning subgraph.)

□