Connectivity and Planar Graphs

- **Cut-vertex:** a vertex v s.t. $G - v$ is disconnected.
- **k-connected:** G remains connected after removal of any $< k$ vertices; $\kappa(G)$ is the largest k.
- **Vertex connectivity $\kappa(G)$:** minimum size of a vertex cut (if none, $\kappa(G) = |V| - 1$).
- **Edge connectivity $\lambda(G)$:** min number of edges whose removal disconnects G.
- **Block:** a maximal 2-connected subgraph (or K_2 or single vertex).
- **Bridge:** an edge whose removal disconnects G.
- **Whitney's Thm:** G 2-connected $\iff$ G has an ear decomposition.
- **Menger's Thm (vertex):** min size of a-b vertex-separator = max number of internally disjoint a-b paths.
- **Menger's Thm (edge):** min edges to separate a-b = max edge-disjoint a-b paths.
- **Menger-Whitney Thm:** G is k-connected $\iff$ $\forall a, b, \exists k$ disjoint a-b paths.
- **Inequalities:** $\kappa(G) \leq \lambda(G) \leq \delta(G)$.
- **Planar graph:** drawable without crossing edges. **Euler's formula:** If G is connected planar, $|V| - |E| + |F| = 2$.
- **Planar bounds:** if $|V| \geq 3$, then $|E| \leq 3|V| - 6$; if no triangle ($girth \geq 4$) then $|E| \leq 2|V| - 4$.
- **Kuratowski:** G planar $\iff$ no subdivision of K_5 or $K_{3,3}$ in G.
- **Wagner:** G planar $\iff$ no $K_5, K_{3,3}$ as a minor.
- **Block composition:** If every block of G is planar, then G is planar (2-sum of planar graphs is planar).
- **Fan Lemma:** In a k-connected G, for any x and any set Y of k vertices, there exist k internally disjoint paths from x to distinct vertices in Y.

Matchings and Factors

- **Matching:** set of edges with no shared endpoints. **Perfect matching:** saturates all vertices. $\nu(G)$ = size of max matching.
- **Saturated/exposed:** vertex is saturated if incident to matching edge; otherwise exposed.
- **Alternating path:** edges alternate between in/out of M. **Augmenting path:** alternating path with both ends exposed.
- **Berge's Thm:** M is maximum $\iff$ no augmenting path (w.r.t. M).
- **Vertex cover:** $W \subseteq V$ s.t. $G - W$ has no edges. $\tau(G)$ = size of min vertex cover.
- **Gallai:** if G has no isolated vertices, $\nu(G) + \rho(G) = |V|$ ($\rho(G)$ = size of min edge-cover). Equivalently, $\alpha(G) + \tau(G) = |V|$ (α max independent set).
- **König:** In bipartite G, $\nu(G) = \tau(G)$.
- **Hall's Thm:** For bipartite $G(A, B)$, a matching saturating A exists $\iff \forall S \subseteq A,\ |N(S)| \geq |S|$.
- **Tutte's 1-factor Thm:** G has a perfect matching $\iff$ $\forall U \subseteq V,\ o(G - U) \leq |U|$, where $o(H) = \#$ of odd components of H.
- **f-factor:** spanning subgraph H with $d_H(v) = f(v)\ \forall v$. (1-factor = perfect matching; r-factor if $f(v) = r$.)
- **Tutte's f-factor Thm:** existence condition (generalizes 1-factor case).
- **Even graph:** all vertices have even degree. A connected graph is Eulerian $\iff$ it is even.
- **2-factor:** 2-regular spanning subgraph (disjoint cycles). **Petersen:** Every d-regular graph with even $d \geq 2$ has a 2-factor.

Vector Spaces of Graphs

- **Characteristic vector:** for spanning subgraph $S \subseteq G$, $w_S \in \{0, 1\}^{|E|}$ with $(w_S)_e = 1$ iff $e \in S$.
- **Cycle (flow) space $W(G)$:** set of w_S for all even subgraphs S (each vertex has even degree in S). $W(G)$ is a $\mathbb{F}_2$-vector space.
- **Dimension:** If G has n vertices, m edges and c components, $\dim W(G) = m - n + c$.
- **Basis (connected G):** Pick a spanning tree T. For each $e \in E \setminus E(T)$, the unique cycle C_e in $T + e$ is a fundamental cycle. $\{w_{C_e}\}$ is a basis of $W(G)$.
- **Cycle decomposition:** Any even subgraph can be expressed (over $\mathbb{F}_2$) as a sum (symmetric difference) of cycles.
- **Girth:** length of shortest cycle in G. In $W(G)$ (binary code), minimum Hamming weight of nonzero vector = girth of G.