Graphs

Graph: $G = (V, E)$, V vertices, E edges.
Types:

- Undirected: $uv \in E$
- Directed: (u, v) arc
- Weighted: c_e for $e \in E$

Degree: $\deg(v) = \#$ incident edges.
Handshaking:

$$\sum_{v \in V} \deg(v) = 2|E|$$

Path: sequence $v_1 v_2, \ldots, v_{k-1} v_k$.
Simple path: $v_i \neq v_j$ for $i \neq j$.
Cycle: closed path.
Connected: $\forall u, v$, path exists.

Cuts

For $U \subseteq V$:

$$\delta(U) = \{uv \in E : u \in U, v \notin U\}$$

s, t**-cut:** $s \in U,\ t \notin U$.
Property: Every s-t path intersects every s, t-cut.

Shortest Path

Given $G = (V, E)$, $c_e \geq 0$, s, t.
Path cost:

$$c(p) = \sum_{e \in p} c_e$$

Goal: minimize $c(p)$.
Special case: $c_e = 1 \Rightarrow$ BFS.

LP Formulation

Primal (SP-P):

$$\min \sum_{e \in E} c_e x_e$$

s.t. flow conservation, $x_e \in \{0, 1\}$

Dual (SP-D):

$$\max \sum_{u \in U} y_u$$

$$\sum_{u : e \in \delta(u)} y_u \leq c_e, \quad y \geq 0$$

Weak duality:

$$c(p) \geq \sum_u y_u$$

Slack

$$\text{slack}_y(e) = c_e - \sum_{u : e \in \delta(u)} y_u$$

Algorithm idea:

- Pick edge with minimum slack
- Increase y_u

Optimization Basics

General:

$$\min / \max \ f(x), \quad x \in A \subseteq \mathbb{R}^n$$

Model components:

- Decision variables x
- Constraints
- Objective function

Linear Programming

Standard Equality Form (SEF):

$$\max c^T x$$

$$Ax = b, \quad x \geq 0$$

Equivalent LPs:

- Same feasibility
- Same boundedness
- Corresponding solutions

Outcomes:

- Optimal
- Unbounded
- Better solution exists

Basic Solutions

Basis B:

- Columns A_B square
- A_B nonsingular

Basic solution:

$$Ax = b, \quad x_j = 0 \ \forall j \notin B$$

Unboundedness

LP unbounded if:

$$\exists r : \ Ar \geq 0,\ c^T r > 0$$

Convexity

f convex:

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

Gradient:

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Subgradient:

$$f(y) \geq f(x) + s^T (y - x)$$

KKT Conditions

For convex g_i and Slater condition:
x^* optimal iff:

$$-c \in \text{cone}(\nabla g_i(x^*),\ i \in \delta(x^*))$$

Tight constraints:

$$\delta(x^*) = \{i : g_i(x^*) = 0\}$$

Slater Condition

x is Slater point if:

$$g_i(x) < 0 \ \forall i$$

Interior Point

Ball:

$$B(x, \delta) = \{y : \|y - x\|_2 \leq \delta\}$$

Interior:

$$\text{int}(S) = \{x : B(x, \delta) \subseteq S\}$$

Key Problems

Shortest Path: polynomial-time solvable.
TSP:

- Visit all vertices once
- Return to start
- NP-hard